Life on the Infinite Farm:

Guide for Parents and Teachers

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Introduction: *Life on the Infinite Farm* is a children's picture book about geometry and infinity. My purpose in writing it was to share my fascination with these topics, which began at a very early age. I tried to make thought-provoking pictures which suggested formal mathematical ideas in an informal way. Mathematicians will recognize these formal ideas, but a young child can enjoy the book without knowing that they are hiding in the background. I think of the infinite farm as a fun, whimsical, and shiny place, but it is lit by the rays of a powerful mathematical sun.

Infinity is one of the oldest and deepest ideas. It is a concept that pervades art, religion, philosophy, and mathematics. In spite of its depth, the concept has very simple conceptual origins. We can imagine processes starting, continuing, then stopping. If we just forget about the starting and/or stopping part, then we confront the infinite. When I was very young I used to think about things like walls or tabletops with no edges, or broomsticks without ends, or fingers that did not have fingertips. Instead of thinking of the compound concept, *finger and tip*, I tried to think of the simpler concept, *finger*, and this lead straight to the contemplation of infinity. I wanted to put these kinds of thoughts into a book.

This guide has two purposes. The first purpose is to help parents read this book and answer questions that their children might have about it. The second purpose is to help teachers use the book effectively in a classroom setting. This guide is divided into five parts.

1. A discussion of the target age for the book.

2. An overview of the book.

3. An explanation of the math behind the book.

4. A suggested simple lesson plan.

5. A suggested advanced lesson plan.

The Target Age: This book has two target ages:

- 5-9
- 10-15

The reason that there are two target ages is that really *Life on the Infinite Farm* is two books in one. I designed the book so that there are two kinds of pages in it, simple and advanced. I marked the advanced pages with little black marks in the corners. (The first page of the book explains the scheme.) The simple pages in themselves tell a complete story. So, a young reader, say ages 5-9, can just skip over the advanced pages. An older reader, say aged 10-15, could read everything together.

The book has about 170 pages in total, and about 50 of them are marked as advanced. So, someone reading just the simple pages will still get a substantial part of the story.

Overview of the Book: The book develops the idea of things which continue without starting and/or stopping in the context of animals on a farm.

Part 1 of the book introduces a number of animals. Here are the three central characters.

1. Gracie is an cow who has infinitely many legs. Her body extends backwards from her head and simply does not end. She loves shoes.
2. Hammerwood is a crocodile who has an infinitely extended mouth. He loves chewing bubble gum.
3. Delores is an infinite squid whose tentacles branch forever. She is the most interesting mathematically. She is modeled on an infinite binary tree. (More on this below.)

The remaining characters, sheep, donkey, gopher, chicken, owl, goat, shark, and a few totally alien-looking creatures, play a more limited role in the book and mostly serve to illustrate some of the ideas in Part 4.

Part 2 of the book introduces problems for each of the main characters:

1. Gracie loves shoes and wears a shoe on every foot. Her friends bring her new shoes for presents and she wants to wear the new shoes without giving up any of the old shoes. How does she do it?

2. Hammerwood loves bubblegum so much that he often forgets to brush his teeth. Many of his teeth fall out, and he has no way to re-grow them. How does he replenish his teeth?

3. Delores wears jewelry on every tentacle, and every weekend her cousin Bin comes over and wants to borrow enough jewelry to decorate his tentacles as well. How can she produce enough extra jewelry so that both she and Bin can decorate their tentacles.

I marked Delores' problem as advanced because her solution is rather intricate. A young child could understand her problem just fine, but perhaps not the solution.

Part 3 of the book explains how the animals solve their problems using their infinite features. The solutions run parallel to the problem.

1. Gracie places the new shoes in front of her and steps forward. Each foot goes into the shoe in front of it. When she is done, she is wearing the new shoes on her front feet. She still wears all her old shoes. Each shoe has just been shifted backwards. This works because there are no shoes left over "at the other and". There is no other end, so to speak.

2. Hammerwood uses powerful rubber bands to compress his teeth back into place. When he starts the process, he has roughly one in ten teeth left. The rubber bands around these remaining teeth contract and remove the gaps between his teeth. When he is done, he has a complete set of teeth. This works because he has an infinite supply to fill in the gaps.

3. Delores doubles her jewelry using a kind of infinite Ponzi scheme. Little fish unclasp her jewelry and bring each ring or bracelet one unit closer towards her head. Each piece moves down one unit and is replaced by two pieces coming in from behind it. Again, this is an advanced part of the book. I will explain the process below in more detail.

The young reader can just read about Gracie's and Hammerwood's solutions and just skip Delores' solution. The role that Delores' problem and solution play in the advanced part of the book is that they highlight the connection between geometry and infinity. What makes Delores' solution so neat is that her particular geometric configuration – that of an infinite binary tree– makes it possible to do what she does.

Part 4: This part of the book elaborates on further features of the infinite farm through a series of questions asked by children and answers given by the narrator. In the simple version of the book, this part is just a fun and whimsical addition to the previous parts of the book. In the advanced version, this part is a more serious discussion of the layout of space and also a very informal introduction to the concept of non-Euclidean geometry. Here are the questions and their answers, with $(*)$ designating an advanced part of the book.

1. (*) How does Delores get her jewelry back from Bin. Answer: They just reverse the process.

2. Where does Hammerwood get his gum? Answer: He points his mouth up an infinite gum tree and just scrapes off what he wants.

3. Can Gracie swing all the way around, like a clock? Answer: No. Her back end would get stuck on all those trees. This is the secret reason why I put in the question about the gum: to highlight the presence of trees on the farm, infinite or otherwise.

4. How do the animals get around each other? Answer: Various ways. The animals have all kinds of adaptations. For instance, they often exist at very different scales, so that one can step right over the other. Sometimes they are very acrobatic and one can roll over another. Delores is built like a flexible puzzle and can temporarily pull herself apart to let other animals through. This is the first really serious question. Having infinite animals moving around presents some serious questions about how they get around each other.

5. (*) How do the infinite animals fit in the pond? Answer. The pond is really a kind of warped non-Euclidean space that is large enough to accommodate infinite animals but still fit on the rest of the farm. This is the most advanced part of the book.

6. (*) Can the farm fit on Earth? Answer: This is subtle. The simple answer is just plain no, because the farm does not close up like the earth. The more advanced answer is that hypothetically the farm could be like one of the warped spaces described in connection with the pond but practically the physical laws of our universe would prevent the farm from actually existing on Earth.

7. (*) Where did Gracie come from? How can the Goat's neck support his infinite horn? etc. Answer: The answer to these various scientific questions is that the most important thing on the farm is geometry, and that the other sciences do not necessarily work the same way. The underlying message is that it is important to focus on geometry when thinking about the farm and not on physical law. The farm is like an idealized Platonic world.

8. Can we visit the farm? Answer: Yes and no. You won't find the farm on Earth but you could imagine it if you read between the lines of geometry books.

The Math Behind the Book:

Gracie's Solution: Gracie's problem and solution is a recasting of the famous Hilbert hotel problem. The Hilbert hotel is a hotel with infinitely many floors that is completely filled with guests. New guests come and want a room, so what does the hotel manager do? He orders everyone to move up one floor. This vacates the bottom floor and the new guests can move in.

The Hilbert hotel scenario, as well as Gracie's version of it, comes down to the mathematical axiom that every positive integer has a successor. In other words, if N is an integer then so is $N + 1$. Applied twice, the axiom says that if N is a positive integer so is $N + 2$.

To get a handle on Gracie's situation, it is useful to label Gracie's feet with counting numbers 1,2,3,4,5,6,... You can think of the left feet as having odd numbers and the right feet as having even numbers. Imagine that white disks represent her Gracie's feet, as shown in Figure 1A. The old shoes are little red disk placed inside the larger white disks, and that the blue dots out in front are the new shoes.

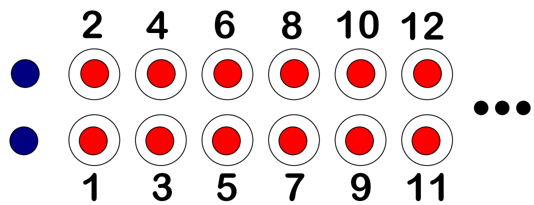

Figure 1A: Old shoes in red, new shoes in blue: before

When Gracie takes a step forward, the shoe in position N moves to the shoe in position $N+2$. So, the shoes in positions $1, 2, 3, 4, ...$ move to the positions $3, 4, 5, 6, ...$ The new shoes occupy positions 1 and 2. Figure 1B shows this.

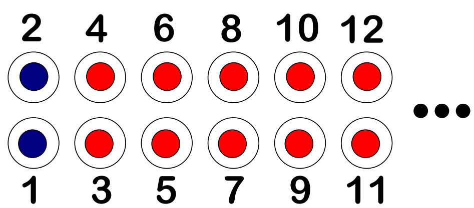

Figure 1B: Old shoes in red, new shoes in blue: after

Hammerwood's Solution: The idea behind Hammerwood's solution is that there is a perfect matching between the positive integers $1, 2, 3, 4, ...$ and the positive multiples of ten, $10, 20, 30, 40, ...$ The matchup is simply $1 \iff 10$, $2 \iff 20$, $3 \iff 30$, etc. Similar to the situation with Gracie, you could imagine that white disks represent the positions in Hammerwood's mouth where *he could have teeth* and the colored disks are placed at locations where he actually has teeth.

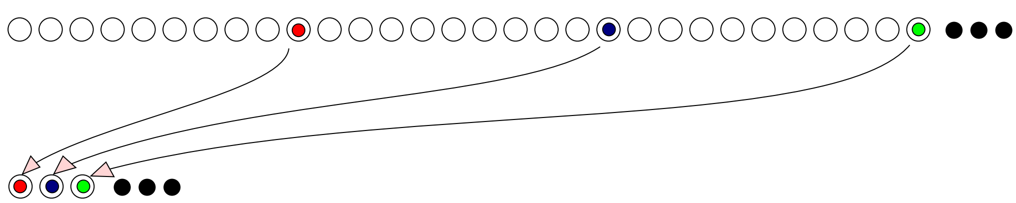
Figure 2: Hammerwood's Teeth, before and after

When the rubber bands contract, the implement the matchup described above, moving the sparsely arranged teeth to the complete set of teeth. The arrows in Figure 2 indicate how this is done.

There is one subtle point about this process. The rubber bands contract to close the gaps between the teeth, just as they would in a real-life situation involving braces, but they need to be anchored at the base of Hammerwoods mouth in order to give a definite direction to the pull. If the rubber bands were not anchored, then, for instance, the first tooth would be pulled forward and who knows where it would land? Incidentally, this would be an excellent question to ask a child or students: Why are the rubber bands tied to a stick?

Delores's Solution: Delores is an infinite binary tree, as shown in Figure 3. Her red spots represent the nodes of the tree and the segments of her tentacles represent the edges. The tentacle segments all have the same length. She wears one piece of colored jewelry per edge. I picture them to be about the length and thickness of a baseball bat.

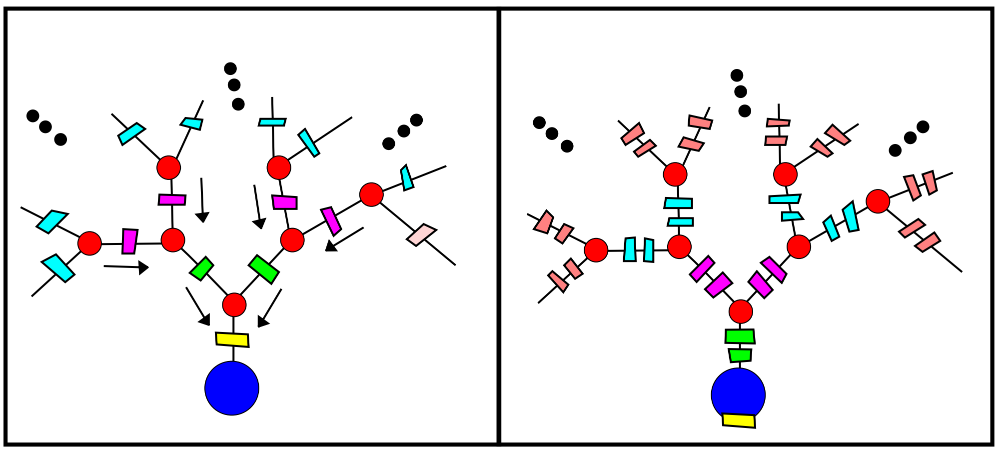

Figure 3: Delores' jewelry, before and after

Delores doubles her jewelry by moving each piece one unit towards her head. This is shown, with the aid of colors, in Figure 3. Since her tentacles branch out forever, she never runs out of jewelry. One fine point about this process is that the piece of jewelry around her neck moves to her forehead and becomes her tiara. Note that the pink jewelry on the right hand side is not seen on the left side; it comes down from further away.

Getting Around Each Other: One nice way to see the problem here is to think about two infinite sharks swim towards each other. The sharks have heads which are essentially infinite lines. Suppose that they swim up to each other and their heads are perpendicular to each other. They would have to rotate so that their heads are parallel in order for one of them to pass the other. There is a similar kind of problem is a squid swims up to a shark. The squid's tentacles could very easily get tangled up around the shark's head.

My solution to this problem is for the sharks to have trap doors in their heads. Parts of their heads can open up like doors to another shark or another kind of animal through. The squids like Delores have a similar adaptation. They are made like flexible puzzles. They can sort of tear themselves apart (temporarily) to let other animals through.

The mathematics underlying these problems has to do with the question of dimensions. In our world, animals are essentially zero dimensional. They are like thickened points. On the infinite farm, the animals are essentially one dimensional. They are like thickened lines, or thickened line-like objects such as trees. Such objects have quite a bit more difficulty moving around each other. So, it is not that surprising that the animals would need to really radical biological adaptations to solve the problem of avoiding each other.

Fitting in the Pond: The sharks and squids would have a lot of problems fitting into the pond if the pond were modelled on ordinary Euclidean space. As I said above, the shark's heads are like infinite lines. Unless these lines are all parallel to the surface of the pond, they will rise and crash through the pond surface. This is the parallel postulate at work.

Figure 4: A tilted shark

The squids have an even more difficult problem. Let's think about the mathematics of Delores. Say that her tentacles are one *unit* long. (As I said above, I think of one unit as being about a yard.) Every time you move one additional unit away from her head, the number of her tentacles doubles. As the units increase, she has $1, 2, 4, 8, 16, 32, 64, \ldots$ tentacle segments. Each one of these segments takes up a fixed amount of volume.

At the same time, let's think about the geometry of 3 dimensional (Euclidean) space. The volume of a solid ball of radius N is πN^3. The problem here is that πN^3 is much smaller than 2^N when N is large. For example

$$\pi(100)^3 \approx 3141592, \qquad 2^{100} = 1267650600228229401496703205376.$$

There is just no way that all that volume could fit inside a ball of radius 100. There is literally too much of Delores to fit into (Euclidean) space.

A student could experience this difficulty first hand. Have the student draw Delores on a big piece of paper so that the edges all have the same length. The students will discover that the picture quickly gets impossibly crowded. The second illustration in the section on fitting into the pond shows this.

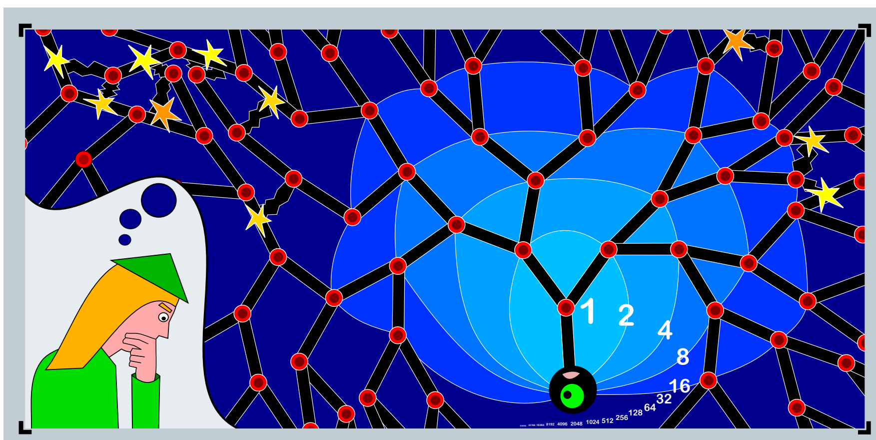

Figure 5: Trying to draw Delores

The solutions to these difficulties is to make the pond a warped non-Euclidean space. One thing to think about here are the famous circle limit pictures by Escher. (For instance, Circle limit I). In Circle Limit I, the bats all have the same size, though they appear to shrink as they approach an ideal circular horizon. If you were to count the number of bats that were within N units of the center, you would discover that there are about 2^N of them. The non-Euclidean world portrayed by Escher is large enough to accomodate Delores' exponential growth. This world is often called the *hyperbolic plane*.

The pond on the infinite farm is essentially a 3-dimensional version of the hyperbolic plane, known as *hyperbolic* 3-*space*. There are some differences between the pond and 3-dimensional hyperbolic space in that part of the pond is accessible from the outside, but these differences are not so important. It is beyond the scope of this guide to describe the precise geometry of the pond. It will probably satisfy students to know that the pond is like a $3D$ version of the world portrayed by Escher in his circle limit series. An extremely bright and interested student could be referred to books on hyperbolic geometry. The most famous of these is probably the book *Geometry and the Imagination* by Hilbert and Cohn-Vossen.

A Simple Lesson Plan: The simple version of *Life on the Infinite Farm* could probably used in a classroom setting, say in a second grade class. Here is how I might use the book in this way.

1. Have a discussion with the students about what infinity means to them? Ask them to picture what it might mean for their arms to go on forever. Would they still have hands and fingers? Give them the example of the positive integers. Ask them to name the largest number and then say "What about your number plus one?" This might lead them to the idea that the integers go on forever.

2. Read the first part of the book. Discuss the different animals and have the students identify what the infinite features are. Ask questions like: "Do you think that Gracie is longer than a freeway?" Have them try to draw pictures of the animals from different points of view.

3. Read about Gracie's problem/solution and ask the children what they would do if they got a new pair of shoes but still wanted to wear the old pair. Have all the kids line up and pretend to be a many legged cow who is faced with a new pair of shoes. Have them enact Gracie's solution and notice that there is a problem with the back end. Have the children draw diagrams of how the solution works.

4. Read about Hammerwood's problem/solution. Have the children make a giant row of marbles (or coins, or checkers) with big gaps between them, and then have them compress the objects to remove the gaps. Have them explain why the compression makes the row get shorter. Try to lead them to the idea that if the row of objects was infintely long, the compression would not chorten the row at all. Informally you could say that one tenth of an infinite length is still an infinite length.

5. Read the various questions about the farm in the book and ask the children if they have any questions of their own. If they don't have any

questions, you could discuss things like: How can Ezekiel hold up his horn? What happens when Gracie eats? Do you think that Gracie ever goes to sleep? Where does she get her shoes? What does the infinite ant colony look like?

6. Ask the children to make up their own infinite animals, and explain the ways in which their creations are infinite.

Advanced Lesson Plan: Here is how I might use the book in, say, an 8th grade class.

1. (As in the simple case) start out with a discussion of infinity. What do students picture when they imagine infinity? Maybe talk about the different ways of representing infinity – e.g. horizon lines, perspective, the (...) construction. Ask then whether they think the universe is finite or infinite? As with the little children, point out the infinite nature of the counting numbers and the integers.

2. Read the first two sections of the book and have the students propose how the animals might solve their problems. Describe the Hilbert Hotel scenario in connection with Gracie's problem. In connection with Hammerwood's problem, explain the notion of a perfect matching between collections of things. In connection with Delores' problem, explain to the students what a binary tree is.

3. Have the students diagram out the animals' solutions to their problems. Probably it would be best to concentrate on Delores' solution, since this is the most advanced. One concrete way that Delores' solution is better than Hammerwood's solution is that most of Hammerwood's teeth would have to be moving faster than (say) the speed of light to actually implement the matchup. In contrast, all the jewelry in Delores' solution moves at the same speed.

4. Have the students attempt to draw Delores on a piece of paper. Try to get them to explain the crowding that they will encounter as a byproduct of exponential growth. After this, have the students look at Escher's circle limit set series and try to understand the way in which the space is warped. Explain that the circle limit is kind like a horizon line. Have the students try to draw a binary tree inside the Escher universe (i.e. the hyperbolic plane) in such a way that all the edges have about the same length and there is no crowding.

5. Have the students discuss the "craters within craters" idea presented in the book. This is the idea that one infinite enclosed space could be fit inside another one, and so on, making for a very complicated farm layout. The farm could have many separate and infinite spaces inside of it. Have the students draw their own diagrams for a layout of craters. In short, have them design their own infinite farm.

6. Have the students discuss some of the biophysical limitations of the farm. How would an infinite animal eat, or hold up an infinite part of its body with a finite amount of muscle. Suggest the idea of massless objects, or maybe different places where gravity focuses.