Notes on the Infinite Farm

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1 Introduction

About 2 years ago, I wrote a rough draft of a book called *Life on the Infinite Farm* as a companion to my book *Gallery of the Infinite*. The idea for *Farm* was to take some of the infinite animals that appeared in *Gallery* and write a story about them that was suitable for younger children. I wrote the whole book in about 4 days and did not spend much time polishing the pictures.

I posted the rough draft on my website and at some point it got about $25,000$ views on tumblr. Some of those many people loved the book and many did not. Some were quite horrified. Others had absolutely no idea what the book was about. A few thought I wrote it while on hallucinogenic drugs.

Recently, after the American Math Society renewed their interest in publishing *Farm*, I reworked the book. I polished the pictures and greatly expanded the last part of the book. The new version is much better, but it also has more sophisticated things in it. (The A.M.S. plans to publish the book, and it should appear around August 2018.)

I wrote these notes to explain exactly what I was thinking about when I wrote the book, page by page. These notes are intended for a sophisticated mathematical audience, but perhaps they will still be of some interest to people who are not mathematicians.

2 Part 1

Page 1: Title.
Page 2: Dedication. “To all my math teachers”.

Page 3: This is a note to the reader which explains how to treat the book as two books in one.

After I drew the animals and showed some of the ways they could use their infinite features to accomplish surprising tasks, I started thinking more about how the animals could actually get around on the farm. I wanted the farm to be realistic, or at least "geometrically plausible". The fact that the animals are infinite in various ways seriously limits their mobility. That made me think more about the geometry of the farm, and this led naturally to the idea that space on the farm was curved in various ways, as it is in hyperbolic geometry. So, I started adding in geometric explanations.

All along I had in mind the idea that the farm was about "geometric infinity", and that is why the last page of the original rough draft has the titles of all kinds of books related to hyperbolic geometry. These later elaborations really did fit in with my original plan, which was mostly unrealized in the draft.

Eventually I had a book that was pretty far from being a children's book. I wanted to have the children's book but I also wanted all this extra stuff. So, I hit on the idea of making two books in one. I arranged things so that there was a kid's book embedded inside the main book. If you just ignore certain pages then what remains still makes sense. That is the kid's version. I think that the target age for this subset of pages is 5-9. If you add back the extra pages, you have the main book. I guess that the main book could be of interest to people of all ages.

Page 4: Opening of Part 1. This is an infinite farmhouse. Gracie the cow is supposed to be inside the farmhouse.

Pages 5,6,7 This is Gracie, an infinite cow. I love cows. Her basic feature is that she has infinitely many legs and an infinite body. Mathematically, she is something like an infinite arc. There is some question about how a single head can power an infinite body. I picture her as having a superconducting nervous system. She is a main character. Gracie's main role is to illustrate the famous Hilbert hotel idea. In the Hilbert hotel, the manager vacates the lowest floor on a full infinite hotel by having all the guests move up one floor.

Page 8: This is Flambeau the sheep and Simon the donkey. These are similar to Gracie mathematically: infinitely many legs, infinite body. Originally

I had imagined Flambeau as billowing outward like smoke. Mathematically speaking, this would be more like an infinite solid cone. I didn't do this because it would have made his habitat from Part 4 much harder to describe. Simon plays a supporting role in the book, giving Gracie gifts and also drinking from the infinite pond. Flambeau serves to illustrate various geometric features of the infinite farm.

Pages 9: This is Gerry the gopher. My daughter Lucy pressed me to make her infinite. One purpose Gerry serves in the book is to illustrate how the infinite pond takes up a bounded amount of space on the infinite farm.

Page 10: This is Ezekiel, an infinite goat. His infinite feature is an infinite horn that is like a logarithmic spiral. There is some issue about how he holds up is horn. I picture it as being massless. Ezekiel plays a minor role in the book, though he appears again on page 74.

Pages 11,12,13: This is Hammerwood the crocodile. He is one of the (unnamed) characters from *Gallery of the Infinite* but he plays a different role there. In *Gallery* he serves to illustrate the bijection between the integers and the natural numbers.

Hammerwood has a new infinite feature: An infinite mouth with infinitely many teeth. There is some question as to how Hammerwood can hold up his head. I picture his mouth as massless. Hammerwood is one of the main characters in the book. He serves to illustrate how division by 10 can be a bijection between a proper subset of the natural numbers to the natural numbers.

What I find most interesting about Hammerwood is that his mouth behaves like an infinite hinge. In a Euclidean world, if he opened his mouth at all it would rise up infinitely high. I imagine that he cannot do this, and so he chews his gum using a kind of grinding, sinusoidal motion. I had been planning to develop this in the questions section, but didn't.

The bubble on page 11 is a genus 2 surface. I had imagined developing Hammerwood further and showing all kinds of exotic bubbles he could blow: higher genus surfaces, knots and links, even infinite surfaces. I didn't take the book in this direction through.

Pages 14,15: This is Penn, the infinite chicken. He is also a (nameless) character from *Gallery*. He appears here almost entirely unchanged. He played the role in *Gallery* that Hammerwood plays in *Farm*, though here the idea is developed more.

At some point, Diana Davis pointed out to me that chickens don't have teeth. However, at that point I had already gotten used to him as he was. He needs teeth in *Gallery* but not in *Farm*. On the other hand, the infinite farm is an alien world with infinite animals. So, it doen't seem to be a stretch to suppose that chickens have teeth on the infinite farm.

Penn's name had been Paddington in the old version. I like Paddington better, but Paddington is also the name of a famous bear in children's lit. I didn't know this when I picked the name; I was thinking of a train station in London.

Page 16: This is Bill, the infinite owl. Bill Thurston was my PhD. advisor at Princeton from 1987-1991 and Bill Goldman was my N.S.F. Postdoctoral mentor from 1994-5 at the University of Maryland. Both these mathematicians have an intense interest in non-Euclidean geometry.

The tiling in Bill's eye is very iteresting to me. At first glance, you might think that it is the famous modular (i.e. Farey) tiling associated to the modular group, drawn in the Poincare disk model of the hyperbolic plane. However, this is not the case. The problem with the true modular tiling (shown left) is that it does not look good when drawn. There is too much crowding of the lines around the cusps, and it takes too many generations to fill in.

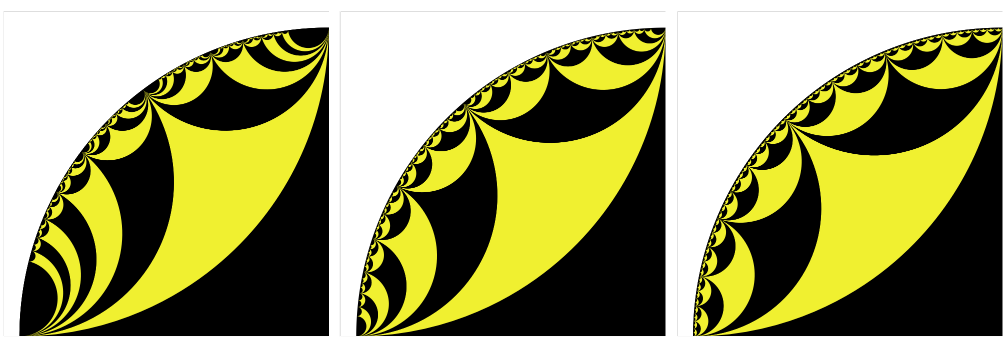

Figure: First 7 generations of 3 hyperbolic tilings.

One fix is to draw a kind of fake modular tiling (shown right) where all the ideal triangles are Euclidean-isosceles. This very nicely fills out the hyperbolic disk and looks pretty too, but it is obviously "fake". What I did is take a kind of interpolation between the modular tiling and stop somewhere in the middle.

Hyperbolic geometers will know what I mean when I say that I took an *earthquake deformation*, a kind of deformation invented by Thurston. However, this is not quite what I did. I based the tiling on an alternate way of averaging fractions that interpolates between Farey addition and ordinary dyadic averaging. This is computationally easier than doing the earthquake and amounts to roughly the same thing.

Pages 17,18: This is Delores, an infinite squid. She is one of the main characters. She is essentially a living, flexible, rooted infinite binary tree. She serves to illustrate another kind of infinite process that is related to non-amenable spaces and their capacity to support Ponzi schemes. I learned of this process from a talk given by Shmuel Weinberger at Berkeley in 1992.

The other important role that Delores plays is that she exhibits exponential growth. She cannot exist in a Euclidean world, because the balls in such a world have polynomial growth. So, the very existence of Delores forces some discussion of hyperbolic geometry – or, more generally, the geometry of metric spaces of nonpositive curvature. A discussion of hyperbolic geometry in turn brings up the sphere at infinity, which is a kind of geometric realization of infinity. The underlying aim of *Farm* is to make this point.

I view *Farm* as a kind of tribute to the beauty of spaces of non-positive curvature and Delores is the device I use. This is why the last page of the book prominently shows the book by Martin Bridson and Andre Haefliger on NPC metric spaces.

Page 19: This is Nelson, an infinite shark. He is friends with Delores. Mathematically, he is basically a rigid straight line (though later on Lily pictures him as folded over.) Nelson plays a small role in the book, but he serves to illustrate what lines are like in various geometries.

Page 20: These are two infinite animals that are alien. Unlike the other animals on the farm, they are infinite due to their fractal nature. I associate the first of these aliens with Dr. Seuss's book *The Cat in the Hat Comes Back*. The second alien is basically a cross between a bird and the Alexander horned sphere.

These aliens don't play any role later in the book, but my point of including them is to indicate that the infinite farm is not "tame" and "predictable" like a farm on earth. There are parts of the farm that are mysterious and even impenetrable to humans.

3 Part 2

Page 21: This is the opener to Part 2. The purpose of Part 2 is to present three problems that the animals can solve using their infinite features. The problems get progressively more subtle and interesting. This page is just filler.

Pages 22,23,24: Gracie's problem is that she wears shoes on every foot and wants to put on new shoes without taking off any of the old ones. This is, of course, impossible for a cow with finitely many legs. For Gracie it is easy. She just shifts all her shoes back and exposes her front legs. This is a variant of the classic Hilbert hotel.

I would call p 24 a collaboration between myself, Eko Hironaka, and Taeer Bar-Yam. Everyone had ideas about how to improve this page.

Pages 25,26,27: Hammerwood's problem is that his excessive gum-chewing causes his teeth to fall out. I imagine that they fall out more or less uniformly, so that after a while he has roughly one out of ten teeth. He can't grow new teeth, and so he needs a way to replenish them. The basic idea is to squeeze his remaining teeth back together. This is essentially the effect of the map $h(x) = x/10$ acting on the subset of natural numbers divisible by 10.

There is some hidden issue about how teeth-rotting bacteria grow on the farm, but I ignore this point and most other biological problems. One reason I mention this is that to me the farm is eternal. The animals are not born and they do not die. The action of teeth-rotting bacteria seems to fly in the face of this.

Hammerwood's problem is similar to Penn the chicken's problem in *Gallery* though in *Gallery* there is no back story about how the teeth are lost. I like the sort of gross way that it happens in *Farm*. I figured that a lot of kids like gross stuff, at least up to a point. I came up with this idea around the time I was contemplating writing and illustrating a comic book about someone who just never brushes their teeth. I never wrote the comic book, but a sliver of it got into *Farm*.

Pages 28,29,30: Delores' problem is that she has to split her jewelry with her cousin Bin (short for Binary) while keeping all her tentacles decorated with jewelry. Basically, I am setting up an infinite Ponzi scheme, the kind that works on a non-amenable space.

Delores' problem could be solved using Hammerwood's trick because, after all, the set of nodes of a rooted infinite binary tree is in bijection here with the natural numbers. However, the significant thing is that Delores is going to solve her problem using *local and geometrically meaningful moves*. No piece of jewelry moves more than 2 units.

4 Part 3

Page 31: This is the opening page of Part 3, where Gracie, Hammerwood, and Delores solve their problems. The picture shows the beginning of what I imagine to be an infinite tractor that scales up linearly as one moves leftward. I was imagining that there would be jobs of all sizes on the farm, and a tractor like this might be useful. However, the tractor plays no role in the book. Basically, I just needed a filler page.

Page 32: This opens Gracie's solution. I have her stand on sticky tape because this seems like a good way for her to pull off her shoes.

Pages 33,34,35,36: Gracie steps out of her shoes and moves her legs one click forward. Her front legs fit inside the new shoes and her other legs fit inside her old shoes.

I deliberately made these pages to be like successive frames in an amination. It would be fun to animate Gracie and her shoe-filling process.

Page 37: End of Gracie's solution. Now she has a new present. I wondered where the shoes come from. I might have asked that question in the Q/A portion in Part 4, but I didn't. My favorite answer is that they flow like lava out of Mount Zeno.

Page 38,39: This opens Hammerwood's solution. It seems sensible that he should clean his teeth first. The brush reminded me of Dr. Seuss.

Page 40: The ants fasten rubber bands around the teeth, so as to squeeze them together. There are a lot of biophysical things wrong with this picture. How would the ants be strong enough to stretch apart the rubber bands? Wouldn't the rubber bands slip off his teeth? I picture the ants as being super strong, or perhaps the rubber bands as having some kind temporal variability. They're easy to stretch during the day and then gain strength at night. As for the slippage, I thought about showing a close-up of Hammerwood's teeth, where you can see that there are little hooks. Or maybe there is enough friction.

Page 41: The first two rubber bands are anchored to a stick. This is very important! Otherwise we are just talking about the affine map $x \to x/10 + C$, where the constant C is not defined. By anchoring the rubber bands, we are setting $C = 0$. So, the final map from depleted teeth to full teeth is $h(x) = x/10$.

As a function of time, the map that is being done is.

$$H_t(x) = \frac{x}{(1-t) + 10t}, \qquad 0 \leq t \leq 1.$$

Note that H_0 is the identity map and $H_1 = h$. In other words, Hammerwood's process is a homotopy between the identity and a contracting map..

I don't imagine this process as being exactly painful for Hammerwood, but rather discombobulating. This crazy process is going on and he is somewhat disoriented during it.

The process really is pretty crazy. The derivative $\partial H_t / \partial t$ is unbounded! So, most of Hammerwood's teeth are moving faster than the speed of light. This would also be the case if Hammerwood could open up his jaw. Indeed, most infinite motions on the farm, anything that involves swinging around, happens at unboundedly fast speeds. The farm is decidedly non-relativistic.

Pages 42,43: This is the end of Hammerwood's teeth fixing process. His teeth are back in place and he is good to go. I imagine the process taking place overnight.

Page 44: This is the start of Delores' jewelry-doubling Ponzi sphere. The first thing the two squids do is line up their tentacles. I have some association here with genetic material coming together for cell division. Diana pressed me to clarify that Delores are both rooted binary trees; hence the little thought bubble showing this.

Pages 45,46: This is the basic jewelry doubling process. Each piece moves one unit towards Delores' head, causing two pieces to fill in at each site where one has moved. This whole procedure needs to be implemented by other animals – small fish in this case – because Delores doesn't have any hands. She can't move the jewelry herself.

Page 47: Something special happens at Delores' neck because the piece of jewelry there strictly speaking cannot move to a tentacle closer to her head. So, this extra piece of jewelry becomes her tiara for the night. Eko pointed out the need to treat the neck specially. My wife Brienne suggested that Delores keep the extra piece on her person as a reminder that she has lent Bin some jewelry. Finally, I thought of the tiara idea.

When geometric group theorists usually consider infinite processes like this, they use formalisms which sweep away small book-keeping problems like this. The notion of a quasi-isometry between metric spaces is a classic example of a fix which takes care of a bounded amount of junk. It is challenging to describe these kinds of processes exactly.

Pages 48,49: This is the end of Delores' solution. Now that Delores has doubled her jewelry she gives half of it to Bin. This is again implemented by little fish.

Page 49 contains an inside joke: Bin thanks Delores for being so *amenable* but it is precisely because she is mathematically non-amenable that she is able to do the trick with the jewelry.

5 Part 4

Page 50: This opens Part 4, a kind of question-and-answer session. This was originally a very small part of *Farm* but I had a lot of fun expanding it out.

In this version, I had originally drawn the kids as frowning or looking totally dazed, but my daughters prevailed upon me to make them look happier. All the kids raising their hands get to ask questions, and a few other kids (not shown) ask as well. I'm pretty sure that it was Eko's idea to involve pictures of children in this part of the book. In the old version I just had the questions.

Page 51: How does Delores get her Jewelry back? This is easy. The process is time-reversable. So, they just "run the movie backwards" so to speak. I had the idea of asking a few softball questions first before getting to the really tricky and interesting questions.

Page 52: Where does Hammerwood get his gum? The little girl Alice imagines that he buys it at a gum shop. Taeer gave me the idea of having Alice be the clerk in the gum shop.

Page 53: Hammerwood gets his gum by sticking his head up an infinite gum tree. The first version of the book had him biting it off an infinite tube which (deliberately) begged the question of where the tube came from. I like this version better because it is more natural. I also liked drawing the trees.

This solution made me confront the idea that the farm would have vertically infinite things. This makes locomotion on the farm more difficult for animals at all scales. I had originally imagined all the trees as finite, so that really tall infinite animals would not see them. The infinite trees are navigation challenges for all the animals that encounter them.

Page 54: Can Gracie swing around like a clock? This question reveals some of the limitations of the animals on the farm.

Page 55: The answer is no. She can't move her infinite back around, say from North to East, because the infinite (or even large finite) trees would get in the way. Instead, she can maneuver her front end around the trees, getting as tangled up as they like.

This picture here is meant to evoke a famous painting by Dennis Sullivan and Bill Thurston on the walls of the Berkeley Math Department in Evans Hall. (Incidentally, Sullivan was another of my postdoctoral mentors.) Their picture shows a simple closed curve wrapped in a very complicated way around three disks. This turned out to be the beginning of Thurston's famous work on pseudo-anosov diffeomorphisms, and more specifically pseudo-anosov elements of the braid group.

My picture doesn't quite show the same thing, because it is a ray. Any two braided rays are isotopic to each other provided that they differ by a finite amount. This is what I mean by saying that Gracie is otherwise very flexible.

I didn't pursue this in the book, but there is actually a way for Gracie to make more radical motions. Imagine that she takes one step back in 1 second, then another step back in 1/2 seconds, then another step back in 1/4 seconds, etc. After 2 seconds, she has disappeared! Where did she go? Worse yet, if time is reversible, then new cows could just show up out of the blue. This a version of Zeno's paradox that has real teeth.

I didn't want to take the farm in this direction because it seems to open up too many issues about what it means to take geometric limits of non-compact spaces. For instance, if Gracie does the Zeno-process stepping forwards rather than backwards, then after two seconds she is an infinite line with no head!

Page 56: How do the animals get around each other? This is the first of the serious questions, and the one that I found the most interesting. Since the animals are infinitely extended, they have much less maneuverability. The preceding question suggests this: Gracie can't turn around in a certain sense.

Page 57: This is the first of the animal tricks. I imagine that the animals occur at vastly different scales, so that the "big" ones can step over the "small ones". This eliminates some of the problems, but still leaves the problem of how the same kinds of animals get around each other.

One could imagine that there are only a few animals at each scale, and then perhaps the maneuverability problem wouldn't be too bad. But, I didn't want to have a farm like this. There should be infinitely many animals of each type!

Page 58: Gracie and Boopis are two cows and they can roll over each other. So, for animals of the same size, one trick is that they are very acrobatic.

There is something hidden here. If Gracie and Boopis are not asymptotically parallel then they will be forever transverse to each other and the rolling trick will not work. So, even though they can twist around like crazy for any finite amount they like, ultimately they have to be pointing in the same direction.

Incidentally, some people have wondered about the funny name "Boopis". This is one of the nicknames of Hera, a greek goddess who could turn into a cow. In short, "Boopis" means "divine cow". I think of the animals as somehow divine. They inhabit a Platonic heaven. (This is coming from an athiest.) I also like words that have "Boop" in them.

Page 59: Nelson the shark's solution is that segments of his head can detach, allowing things to pass through it. This seems like a wierd adaptation, but something radical has to happen. How does he stay alive during this process? I don't worry about that. You could imagine that Nelson is really some kind of infinite colony, made of smaller living units, but you could also imagine a more alien kind of biology that supports detaching parts.

Pages 60: Delores has a related kind of solution. On a small scale, she is like a living puzzle. She can tear herself apart and let other things through. I had originally done more with this, where I talked about the squids switching body parts – mixing and matching. But I decided to cut this out because it I felt there was already so much about Delores.

It is worth emphasizing that the animals do not need these tricks from a topological point of view. It isn't like they get linked and then can't get unlinked without cutting. These cut and paste tricks are for *geometric convenience*. It is not topologically impossible to walk from Providence to Los Angeles, but it is quite geometrically inconvenient. So, these tricks are convenient in the same way that an airplane is a convenient way to get across the country.

Page 61: This page starts the geometric exploration of the farm. I decided to start with the geometry of the pond, though really I am interested in the geometry of the land as well.

The girl in the picture is basically my daughter Lily. She is contemplating various ways that the infinite animals would have problems fitting in an infinite pond. She starts out with a Euclidean understanding of space.

The shark, Nelson, is essentially a straight line. He would have trouble fitting in various kinds of subsets of $\boldsymbol{R}^3$, even if they are infinite. Lily imagines the pond on the left to be something like the area under a down-turning parabola, but with an opening at the top. No line can fit underneath a down-turning parabola. This is why she pictures Nelson crashing into the side of the pond.

Of course, Nelson would fit just fine in a pond that was a half-space, or even an infinite solid cylinder. However, thanks to the parallel postulate, Nelson would have quite a bit of trouble tilting his head. He would always have to stay parallel to the water surface, or else some part of his head would crash through it.

P 62: The problem with Delores fitting into the pond is both more subtle and more severe. Delores has 2^N tentacles within N units of her head. These tentacles all have the same size. Say they are all like a person's leg. The problem is that the volume of a 3-dimensional ball of radius N is about N^3. Since N^3 is much smaller than 2^N for large N, Delores will inevitably experience too much compression to fit in the space. This happens no matter how thin and wispy her tentacles, as long as they have a uniform thickness. She simply does not fit in Euclidean space.

Delores would have the same fitting problem in a Euclidean space of any dimension. There is nothing special about 3-dimensions here. Indeed, Lily imagines a 2-dimensional cartoon version of this problem. The various stars in the picture indicate where Delores cannot fit. I think that the 3D version of this would be too hard to draw.

I first saw this issue explicitly discussed in the book *Random Walks and Electric Networks*, by Peter Doyle and Laurie Snell. They talk about the concept of being able to gracefully draw a graph in Euclidean space, and a rooted binary tree cannot be gracefully drawn.

Page 63: After mulling it over for a while, Lily finally asks how the animals can fit in the pond. She pictures Delores and Nelson as shooting out of the pond, presumably because the pond is too small to contain them.

Page 64: This page presents the main idea. The pond is like a copy of 3-dimensional hyperbolic space. The only difference is that the top of the pond is "incomplete", so that Simon the donkey can still drink from it. The whole pond fits in a finite part of the infinite farm from the outside, so Gerry the gopher can bang his head against it from the outside, and tunnel underneath it.

Mathematically, I imagine that there is a discontinuous Riemannian metric ρ on $\boldsymbol{R}^3$ with the following properties:

- ρ agrees with the hyperbolic metric on a subset of hyperbolic space which contains a hyperbolic half-space.
- ρ agrees with the Euclidean metric on the complement of a bounded open set. In particular, this Euclidean subset is closed. The frontier of the metric is the interface between the Euclidean and hyperbolic parts.

- ρ is discontinuous at the interface just mentioned. This interface is a spherical cap. Gerry bangs his head on the interface.

Notice that ρ is defined on a connected subset which includes both hyperbolic and Euclidean pieces. So, there is communication between these two very different worlds.

The presence of hyperbolic geometry solves all the problems that Lily worries about on pages 61-62. Hyperbolic space supports isometric embeddings of rooted binary trees. There is so much room in hyperbolic space that one can draw such a tree in an undistorted way.

The hyperbolic geometry also deals with the parallel postulate. The sharks can be in lots of different positions relative to the water-air interface and not stick out of it.

Page 65: This illustrates the hyperbolic pond idea on land. Mathematically there is a Riemannian metric like last time, except now it agrees in the crater with the product of the hyperbolic metric and an interval. (That is, locally the geometry is that of $\boldsymbol{H}^2 \times \boldsymbol{R}$.) This allows Flambeau and the other sheep to stand on the bottom of the crater and yet have the sides of the crater infinitely far away. The crater has non-positive rather than negative curvature, though in the important direction it is negatively curved.

The names of the other sheep are significant here. Bolyai, Lobachevski, and Gauss are the three (independent) co-discoverers of hyperbolic geometry.

The metric around the top of the crater is interesting. You have a finite bottleneck that opens up into an infinite space. I picture that the crater is like a huge tuna can with a pinhole poked into the top of it.

Page 66: Here is a close-up of Bill. One unresolved issue from Part 1 is the nature of Bill's infinite aspect. Unlike the other animals (except the aliens on Page 18) Bill seems to have a kind of fractal aspect rather than an infinitely extended one. The resolution here is that Bill's eye is another one of these enclosed infinite spaces, so actually all the pupils in his eye are infinitely large. The fractal nature comes from an outsiders perception of the warping of space. The geometry of his cornea is essentially the same as a cross-section of the crater: the hyperbolic plane. That is why I say that Bill can see the crater perfectly.

It is interesting to think about how Bill sees the crater. Imagine a Riemannian metric on an open cylinder which imparts the following features to the cylinder:

- It has radial symmetry.
- There is an isometry which swaps the top and bottom of the cylinder.
- The central cross section of the cylinder is isometric to a Euclidean disk.
- The top and bottom are isometric to the hyperbolic plane.

The top of the cylinder is Bill's eye. The bottom is the bottom of the crater. Any geodesic which starts out at the central cross section perpendicular to it will hit the top and bottom of the cylinder in corresponding points. You can interpret such a geodesic as a beam of light going from the bottom of the crater to Bill's eye. Every point on the bottom of the crater has such a light beam emanating from it.

What is going on is a radical compression and then decompression of visual data. All that light from the infinite crater squeezes down onto a finite disk – the central cross section – and then expands out again into the infinite eye. This kind of vision is incomparible with the granular (i.e. cellular and atomic) nature of animals on Earth, but it is OK on the farm.

Page 67: Just as the infinite farm has a crater in it, it could happen that the crater has a smaller crater in it. This would allow for multiple infinite habitats within the infinite farm.

One alternate solution I had is to say that the infinite farm has many disconnected components. The trouble with this approach is that many readers will balk at the idea of a space having separate and inaccessible components that are not embedded in a larger space. I think that this question tied up philosophers for centuries, because they couldn't picture a finite universe without a boundary.

Another idea would be to compromise and have the infinite farm be a bunch of slices of a higher dimensional space. My favorite idea here is to have the farm be a union of totally geodesic hyperbolic 3-spaces sitting inside hyperbolic 4-space in a pattern that is symmetric under the action of a 4-dimensional hyperbolic lattice.

Page 68: The observation from Page 67 can be iterated. There can be craters within craters within craters. This is another kind of geometric infinity: an infinite nesting of craters. Mathematically this corresponds roughly to the free product of two hyperbolic groups - e.g. the fundamental group of the connected sum of two hyperbolic manifolds.

Page 69: The existence of craters within craters leads naturally to the question of whether the farm can exist on Earth. The girl in the picture is my daughter Lucy. She imagines Flambeau shooting off Earth like a tangent ray.

Page 70: The first answer to the question is an obvious no! Since the farm is infinite it would seem that it could not fit on the finite earth. The picture shows Earth as a kind of small plaything for Gracie and Hammerwood. I didn't actually fix the scales for these animals, but this page suggests that e.g. Gracie's head is about as big as the earth.

Page 71: This this the set-up for a more subtle answer to the question. I might have had Pages 69-70 be a single page, but I wanted the sequence 69,70 to make sense as a stand-alone narrative, with 71,72,73 being an enhancement.

Page 72: A more subtle answer to the question is a possible yes. Given the theoretical possibility of craters embedded in craters, why not have this possibility on Earth?

Page 73: An even more subtle answer is no again! These kinds of craters would be prevented by physics. This is an interesting and subtle issue. The craters are warped space and our universe certainly contains some warped space - e.g. black holes. The black holes are supposed to have singularities, and this would be akin to the infinite warping needed for hyperbolic space. A black hole is a kind of infinity embedded in a finite space. I imagine that the theoretical singularity of the black hole corresponds to the ideal boundary of hyperbolic space.

One theoretical problem is that this infinite warping is not really compatible with the granular (i.e. atomic) nature of matter. Mathematically, Riemannian manifolds are modeled on $\boldsymbol{R}^n$, a space that looks the same at all scales. On the other hand, our space is subject to various constraints coming from quantum mechanics. Things look very different at small scales. You have things like Planck's constant and the Heisenberg Uncertainty Principle.

One practical problem is that warped space like this would probably need to coexist alongside an infinite gravitational field. I guess that this is a consequence of the Einstein field equations: The curvature tensor (which encodes the warping of space) equals the stress energy tensor (which encodes the presence of matter and energy.) There isn't enough matter on Earth, or indeed in the whole of our universe, to support one of these craters. Put another way, the gravitational field associated to such a crater would crush everything around it.

I'm not really an expert on mathematical physics. This is just my best guess as to what is going on. I put some explanation like this in a footnote. This way, the disinterested reader can skip it. I hope that the footnote makes it clear that this explanation is meant for older, more advanced readers.

Page 74: How does day change to night on the farm? This is another one of these physics-type questions that I don't really know how to answer.

I had in the back of my mind the kind of objections one might have to the flat-Earth theory: What happens at sunset? Does the sun just crash into the ground or into the ocean surrounding the Earth.

Page 75: My first answer to this question is to just imagine that the suns and moons are like lights on dimmers. They change their brightness and color to suit the time of day. I draw the lights in a way that suggests a Euclidean grid, but given all the space warping on the farm, I really have in mind a much more complicated layout of lights. One could imagine the lights being distributed over a crater according to a hyperbolic lattice.

Page 76: My second answer is more subtle. Imagine first that the farm is like an infinite cylinder. It would be infinite in one direction and yet suns and moons could revolve around it. Here I have something in mind in higher dimensions: The earth could be the product of a plane and a disk and the ambient space could be 4-dimensional. The suns and moons could then revolve around the farm without hitting it.

I tried to evoke this idea by drawing the infinite animals as living in two-dimensional slices, while three dimensional suns and moons (represented as cubes) sort of move around it. I thought it would be too hard to evoke the fourth dimension more directly. There is a famous painting of a "4D cross" by Salvidore Dali, but this never really convinced me. There is also the famous *Nude Descending the Stairs* by Marcel Duchamp. Better yet, I might like to invent my own way of depicting 4-dimensional solids in a convincing way. I have no idea how to do it.

Page 77: My third answer is supposed to be nonsensical but poetical. My wife Brienne came up with it. The idea is that an infinite flock of glowing birds brings the light to where it is needed. The underlying point of this page is that I actually don't know how light comes to the farm (though I stop short of saying this, to keep an omniscient narrator.)

Do we really know why we have light in our universe? Or matter? Or anything else? We simply give names to certain kinds of sense-data and we make mental models which have more or less success at predicting how this sense-data behaves. (I pretty much believe what David Hume says about the nature of our understanding of physical law.) As for the underlying questions – e.g. *What is a photon really?* – we have no idea. The photons might as well be glowing birds for all we actually know.

My wife and I had some other wacky ideas along these lines. Another idea involved glowing volcanos distributed all over the farm. My favorite was that the light was carried around by ostriches with glowing heads, and they would stick their heads in the ground sometimes and that would temporarily extinguish the light.

Page 78: This page is a clearing house for similar kinds of hard-to-answer scientific questions. Originally I had many more, but I pared it down to 4. I got some inspiration from the 4-questions in the Jewish passover service. In the service there are 4 types of kids and each asks a question which reflects his temperament. In my case, the 4 kids grow older and more sophisticated, and so do their questions. The 4 kids also represent some abstract categories for me:

- philosophy
- biology
- chemistry
- physics

I drop the philosophy theme in subsequent pages and just concentrate on the trio of bio-chem-physics. Or maybe my character carries on the role of philosopher.

Incidentally, here are some questions I had on my list but didn't ask:

- Does Gracie eat? Where does the food go?
- How does self-similarity interact with the Heisenberg uncertainty principle?
- How does the nitrogen cycle work on the farm?
- Did the farm have a big bang?
- Is there gravity on the farm?
- How can Penn the chicken balance on 2 legs?

Page 79: This page is the start of my catch-all answer to these kinds of science questions: the laws of science do not really apply on the infinite farm. Again, the infinite farm is an idealized Platonic world designed to accommodate geometric ideas.

I pictured myself as being skewered by an "impossible cube", a kind of optical illusion that doesn't exist in 3D space. My impossible cube has 3 colors, corresponding to the colors of the personified sciences. So, symbolically, I am being skewered by the sciences.

I had originally said "I don't know" to these questions, but Eko and I eventually felt that it was better to have the narrator retain his omniscience.

Page 80: This was one of my favorites to draw. I like the ideas of the sciences as being personified. In our universe they are relentless enforcers. They control everything. I imagine that the sciences exist on the infinite farm but they have a weak influence. They kind of fool around with laws in a half-hearted kind of way but don't really enforce them very often.

The picture shows three kinds of games:

- Bio is making DNA codons while watching the DNA-like spiral of Ezekiel out the window. The DNA codon shown, UAG, is the codon for the *stop* instruction on DNA. I had in my mind that the bio guy was trying in a very weak and roundabout way to tell Ezekiel to stop fooling around outside.
- Chem is making the beginning of the periodic table.

- Phys has spelled out $E = mc^2$.

Page 81: The sciences take a backseat to geometry, which is really the ruler of the infinite farm. Here I tried to evoke the idea that the reality of the farm springs out of our geometric ideas and diagrams. So, the 3D picture of Hammerwood and Ezekiel is supposed to reflect the 2D plan that is drawn on the sketch pad.

Page 82: One last question: Can we visit the farm? I liked the idea of the kids being carried there by an infinite school bus. This doesn't really make sense, because there are only finitely many kids on Earth, but I liked the image. Also, I was imagining that maybe the farm is common property of like-minded beings on infinitely many (potential) worlds. This goes along with the commonly held belief that a large part of mathematics would be the same in any universe with sentient life. I'm not sure I believe this, but that is what I am thinking about here.

Page 83: This is the start of my answer to the last question. The answer: not really. The pic shows me standing in front of the library in a complex of buildings that could be a university or a research institute. To me the place has the same sort of feel as the Institute for Advanced Study in Princeton. I am drawing myself a bit younger than I actually am.

When I drew the picture, I was thinking about my sabbatical at I.A.S. in 2004, which was a really fabulous year for me – pretty much the happiest time in my life. I tried to capture that serene feeling that comes from having an ocean of time to pursue whatever interests you.

Page 84: The end: Here I am standing in the library and then finally sitting at a desk in the library reading the book by Bridson and Haefliger. There are other books on the shelf as well. The ones with capital letters are real books:

- *The Geometry and Topology of* 3*-Manifolds*. These are Thurston's famous 1978 notes. They were subsequently published in book form in the 1990s'.

- *Geometric Group Theory*, by Cornelia Drutu and Misha Kapovich. This book has a ton of material about all kinds of infinite spaces. When I first thought about writing *Farm* I was on sabbatical in Oxford and had just met Cornelia's baby, Alexandra. I got the idea of writing a book that professional mathematicians and their children could enjoy at the same time.

- *Geometry and the Imagination* by David Hilbert and S. Cohn-Vossen. This is classic text about geometry.
- *The Cat in the Hat Comes Back* by Dr. Seuss. This is one of the most mathematical children's books ever written.
- *Gallery of the Infinite*. For obvious reasons.
- *Really Big Numbers*. This is another of my books.

The final sentence in the book is supposed to suggest the hidden and scholarly origins of the infinite farm, as if it were literally powered by books and ideas. I tried to make the infinite farm a shiny and whimsical place, but it has deep mathematical underpinnings.

Page 85: Back page with info about me and acknowledgements of various kinds.