The Optimal Paper Moebius Band

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Abstract

We prove that a smooth embedded paper Moebius band must have aspect ratio greater than $\sqrt{3}$. We also prove that any sequence of smooth embedded paper Moebius bands whose aspect ratio converges to $\sqrt{3}$ must converge, up to isometry, to the famous triangular Moebius band. These results answer the mimimum aspect ratio question discussed by W. Wunderlich in 1962 and prove the more specific conjecture of B. Halpern and C. Weaver from 1977.

1 Introduction

To make a paper Moebius band you give a strip of paper an odd number of twists and then tape the ends together. For long strips this is easy and for short strips it is difficult or impossible. Figure 1 shows a famous example called the *triangular Moebius band* that is based on a $1 \times \sqrt{3}$ strip.

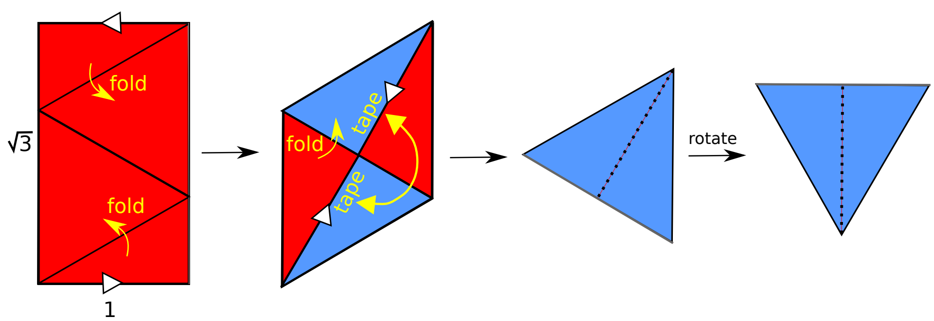

Figure 1: The triangular Moebius band

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The strip in Figure 1 is colored red on one side and blue on the other. You are supposed to fold and somehow tape the thing as indicated in Figure 1. The tape runs along the dotted line in the "inside" of the little triangular "wallet" you are making. The final rotation highlights a kind of "T-pattern" made from the top edge and the dotted line, a pattern that is important in this paper. You might enjoy finding other ways of making this example in which the taping is easier to manage.

What is the smallest λ for which we can turn a $1 \times \lambda$ strip into a paper Moebius band? In order to answer this question we have to be more formal about what we are doing. Formally speaking, a *smooth paper Moebius band of aspect ratio* λ is a smooth isometric mapping $I : M_\lambda \to \boldsymbol{R}^3$, where M_λ is the flat Mobius band

$$M_\lambda = ([0,1] \times [0,\lambda])/\sim, \qquad (x,0) \sim (1-x,\lambda) \tag{1}$$

An *isometric mapping* is a map whose differential is an isometry. The map is an *embedding* if it is injective, and an *immersion* in general. The image $\Omega = I(M_\lambda)$ is an example of a developable surface (with boundary). I learned about paper Moebius bands from the beautiful expository article [**FT**, Chapter 14] by Dmitry Fuchs and Sergei Tabachnikov.

The early papers of M. Sadowsky [**Sa**] and W. Wunderlich [**W**] treat both the existence and differential geometry of smooth paper Moebius bands. (See [**HF**] and [**T**] respectively for modern English translations.) The paper [**CF**] gives a modern differential geometric framework for smooth developable surfaces.

Why bother with smooth maps? Well, if you just look at ways of folding paper up to make a Moebius band you can get all kinds of weird examples. For instance, you could take a square, fold it like an accordion into a thin strip, twist, then tape. This monster is not approximable by smooth examples. The smooth formalism rules out pathologies like this. in contrast, the triangular paper Moebius band can be approximated to arbitrary precision by smooth embedded paper Moebius bands. See [**Sa**], [**HW**], and [**FT**].

W. Wunderlich discusses the minimum aspect ratio question in the introduction of his 1962 paper [**W**]. He says that it is easy to make a paper Moebius band when $\lambda \geq 5$ and that the minimal value is not known. Since it is a very natural question I can imagine that it had been raised even earlier, but I don't know where or when.

In their 1977 paper [**HW**], Halpern and Weaver study the minimum aspect ratio question in detail. They prove two things.

- For immersed paper Moebius bands one has $\lambda > \pi/2$. In particular, this bound holds for embedded paper Moebius bands. Moreover, one can find a sequence of immersed examples with the aspect ratio converging to $\pi/2$. These examples are not embedded.
- One can find a sequence of embedded paper Moebius bands with the aspect ratio converging to $\sqrt{3}$. These examples converge to the triangular Moebius band. (The triangular Moebius band itself does not count as an embedded smooth paper Moebius band.)

The last line of [**HW**] states the conjecture that $\lambda > \sqrt{3}$ for an embedded paper Moebius band.

In this paper I will prove the Halpern-Weaver Conjecture and show that the triangular Moebius band is uniquely the best limit.

Theorem 1.1 (Main) *A smooth embedded paper Moebius band has aspect ratio greater than* $\sqrt{3}$*.*

Theorem 1.2 (Triangular Limit) *Let* $I_n : M_{\lambda_n} \to \Omega_n$ *be a sequence of embedded paper Moebius bands such that* $\lambda_n \to \sqrt{3}$*. Then, up to isometry,* I_n *converges uniformly to the map giving the triangular Moebius band.*

The work here supersedes my earlier paper [**S**] and also is independent from it, but nonetheless it is an outgrowth of [**S**]. In [**S**] I improved the lower bound $\lambda > \pi/2$ in the embedded case to a bound $\lambda > \lambda_1$ for some complicated number $\lambda_1 \in (\pi/2, \sqrt{3})$. Let me explain this in some detail.

An embedded paper Moebius band $\Omega = I(M_\lambda)$ has [1] a continuously varying decomposition into straight line segments having their endpoints in the boundary. We call these segments the *bends*. They are sometimes called the ruling lines. (We call the corresponding pre-images of these bends on M_λ the *pre-bends*.) We say that a *T-pattern* is a pair of bends which lie in perpendicular intersecting lines. We call the T-pattern *embedded* if the two bends are disjoint. In [**S**] I proved the following result under the additional hypothesis that $\lambda < 7\pi/12$, and in the embedded case.

Lemma 1.3 (T) *A paper Moebius band has a* T*-pattern.*

[1] This is a classic result. See e.g. the bottom of p. 46 in [**HW**]. Here they explain that the subset of points of non-zero mean curvature on a paper Moebius band has a foliation by such segments. One can then extend the foliation continuously to the complement.

I deduce easily from the version of Lemma T in [**S**] that $\lambda \geq \phi$, the golden ratio. (This amounts to taking $t = 0$ in Equation 2 below.) Then I solve an optimization problem to get a lower bound $\lambda_1 \in (\phi, \sqrt{3})$. Embarrassingly, I discovered recently that I made an error in setting up the optimization problem in [**S**]. I mistakenly assumed that when you cut M_λ open along an embedded line segment which joins two points in ∂M_λ you get a parallelogram rather than a trapezoid. This idiotic mistake caused me to miscalculate λ_1. So, all I can conclude from [**S**] is that $\lambda_1 \geq \phi$. However, I was amazed and delighted to discover that when I did the optimization problem correctly I got $\lambda_1 = \sqrt{3}$ right on the nose! Here is the form the optimization calculation takes in this paper.

Lemma 1.4 (G) *A paper Moebius band with an embedded T-pattern has aspect ratio greater than $\sqrt{3}$.*

Here the paper Moebius band need not be embedded; we are only insisting that the T-pattern itself be embedded. Of course, a T-pattern on an embedded paper Moebius band is embedded. The Main Theorem follows immediately from Lemmas G and T.

The topic of paper Moebius bands is adjacent to a number of different subjects. The paper [**GKS**] considers the related question of tying a piece of rope into a knot using as little rope as possible. See [**DDS**] for further results. One could view these rope knot questions as variants of the Halpern-Weaver Conjecture in a different category. Indeed, our Lemma T seems quite related in spirit to the quadrisecent idea in [**DDS**].

Paper Moebius bands are even more closely related to *folded ribbon knots*, and the triangular Moebius band can be interpreted as a folded ribbon knot. See [**D**] for a survey on this topic. More precisely, see [**DL**, Corollary 25] for a result which is in some sense a special case of our two results and see [**DL**, Conjecture 26] for a variant of the Halpern-Weaver Conjecture in the category of folded ribbon knots. I will say more about this in §5.3.

Some authors have considered "optimal Moebius bands" from other perspectives. The papers [**Sz**] considers the question from an algebraic perspective and the paper [**MK**] consider the question from a physical perspective. See also [**SH**].

This paper is organized as follows. In §2 and §3 respectively I will give short and self-contained proofs of these results. (The side hypotheses for Lemma T that I had in [**S**] are not needed for the new proof.) The proof of the

Triangular Limit Theorem, given in §4, amounts to examining what our proof of Lemma G says about a minimizing sequence. In §5 I will include some remarks about the topics in this paper. Most of these remarks are inspired by my conversations with other mathematicians who have read earlier drafts of this paper.

Here is one more thing I'd like to mention. Some readers might find this paper hard to read because I do not include much background information. I have subsequently written a longer and friendlier account [**S2**], aimed at university students and perhaps advanced high school students. This paper is available on my Brown University website.

I thank Matei Coiculescu, Robert Connelly, Dan Cristofaro-Gardiner, Elizabeth Denne, Ben Halpern, Dmitry Fuchs, Javier Gomez-Serrano, Anton Izosimov, Jeremy Kahn, Stephen D. Miller, Noah Montgomery, Sergei Tabachnikov, and Charles Weaver for helpful discussions about this subject. I especially thank Matei for suggesting that I try for a "mapping proof" of Lemma T as opposed to the kind of proof I had previously. That suggestion led me to find a really nice proof of Lemma T that greatly simplified this paper.

2 Proof of Lemma G

Let ∇ be a triangle with horizontal base. Let $p(\nabla)$ be the perimeter of ∇ and let $n(\nabla)$ be the sum of the lengths of the non-horizontal edges of ∇.

Lemma 2.1 *If ∇ has base $\sqrt{1+t^2}$ and height $h \geq 1$ then $n(\nabla) \geq \sqrt{5+t^2}$ and $p(\nabla) \geq \sqrt{1+t^2} + \sqrt{5+t^2}$. Equality occurs iff ∇ is isosceles and $h = 1$.*

Proof: This is an extremely well known kind of result. Let $\beta = \sqrt{1+t^2}$.

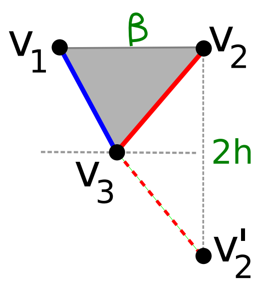

Figure 2.1: The diagram for Lemma 2.1.

Let v_1, v_2, v_3 be the vertices of ∇, with v_3 the apex. Let v_2' be the reflection of v_2 through the horizontal line containing v_3. By symmetry, the triangle inequality, and the Pythagorean Theorem,

$$n(\nabla) = \|v_1 - v_3\| + \|v_3 - v_2'\| \geq \|v_1 - v_2'\| = \sqrt{\beta^2 + 4h^2} \geq \sqrt{\beta^2 + 4} = \sqrt{5+t^2}.$$

The bound for $p(\nabla)$ follows immediately. In the case of Equality, $h = 1$ and v_1, v_3, v_2' are collinear, meaning that ∇ is isosceles. ♠

Let $I : M_\lambda \to \Omega$ be a paper Moebius band with an embedded T-pattern. We write $S' = I(S)$ for any relevant set S and we let $\ell(\cdot)$ denote arc-length. By definition, we have $\ell(\gamma) = \ell(\gamma')$ for any curve $\gamma \subset M_\lambda$. For instance, $\ell(\partial M_\lambda) = \ell(\partial \Omega)$.

Let B' and T' be the pair of disjoint bends comprising an embedded T-pattern of Ω. Since they lie on intersecting lines, B' and T' are co-planar. We choose so that the line extending T' is disjoint from B', then rotate so that B' and T' are respectively vertical and horizontal segments in the XY-plane and B' is strictly below the line extending T'. Let B and T be the pre-bends corresponding to B' and T'. We cut M_λ open along B to get a bilaterally symmetric **trapezoid**. See Figure 2.2.

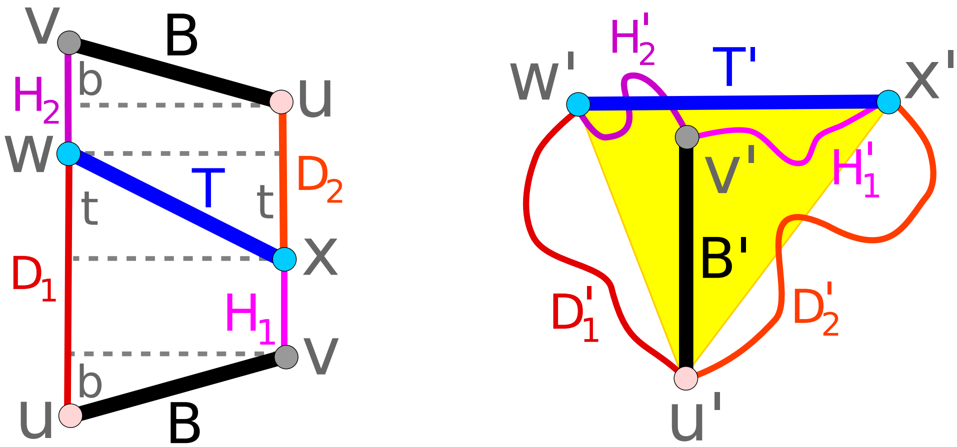

Figure 2.2: The trapezoid (left) and the T-pattern (right).

Here $-t$ is the slope of T. The quantity b, which is the slope of the bottom choice of B, plays no role in our calculations. The picture looks a bit different when the signs of t and b are different, but it is always true that $\ell(H_1)+\ell(H_2)=\ell(D_1)+\ell(D_2)-2t$. The yellow triangle ∇ has base $\sqrt{1+t^2}$ and height greater than 1.

First Bound: We have $2\lambda > \sqrt{1+t^2}+\sqrt{5+t^2}$. Here is the derivation:

$$2\lambda = \ell(\partial M_\lambda) = \ell(\partial\Omega) \geq p(\nabla) > \sqrt{1+t^2}+\sqrt{5+t^2}. \tag{2}$$

The first inequality comes from the fact that $\partial\Omega$ is a (red and magenta) loop containing all vertices of ∇. The second inequality is Lemma 2.1.

Second Bound: We have $2\lambda > 2\sqrt{5+t^2}-2t$. Here is the derivation.

$$2\lambda = \ell(D_1)+\ell(D_2)+\ell(H_1)+\ell(H_2) = 2\ell(D_1)+2\ell(D_2)-2t =$$

$$2\ell(D_1')+2\ell(D_2')-2t \geq 2n(\nabla)-2t > 2\sqrt{5+t^2}-2t. \tag{3}$$

The first inequality comes from the fact that $D_1' \cup D_2'$ is a (red) path that connects w' to x' and contains u'. The second inequality is Lemma 2.1.

Combining the Bounds: Let $t_0 = 1/\sqrt{3}$. If $t \geq t_0$ then our first bound gives $\lambda > \sqrt{3}$. If $t \leq t_0$ then our second bound gives $\lambda > \sqrt{3}$. Hence $\lambda > \sqrt{3}$. This completes the proof of Lemma G.

3 Proof of Lemma T

Let $I : M_\lambda \to \Omega$ be a paper Moebius band. We only care about the embedded case but the proof works in all cases. We parametrize the space of pre-bends of M_λ with the circle $\boldsymbol{R}/\lambda\boldsymbol{Z}$ as follows: Each pre-bend corresponds to its intersection with the centerline of M_λ, and this is a copy of $\boldsymbol{R}/\lambda\boldsymbol{Z}$. Using I, we simultaneously parametrize the bends of Ω by $\boldsymbol{R}/\lambda\boldsymbol{Z}$.

The Cylinder: Let Υ be the cylinder obtained from $(\boldsymbol{R}/\lambda\boldsymbol{Z})^2$ by deleting the diagonal. A point $(x, y) \in \Upsilon$ corresponds to a pair (u, v) of bends. We let $\overline{\Upsilon}$ be the compactification of Υ obtained by adding 2 boundary components. The point (x, y) lies near one boundary component if y lies just ahead of x in the cyclic order coming from $\boldsymbol{R}/\lambda\boldsymbol{Z}$. The point (x, y) lies near the other boundary component if y lies just behind of x in the same cyclic order. Let $\partial\overline{\Upsilon}$ be the boundary of $\overline{\Upsilon}$.

Oriented Bends: Let $(x, y) \in \Upsilon$ be arbitrary. There is a unique minimal path $x_t \in \boldsymbol{R}/\lambda\boldsymbol{Z}$ such that $x_0 = x$ and $x_1 = y$ and x_t is locally increasing with respect to the cyclic order on $\boldsymbol{R}/\lambda\boldsymbol{Z}$. This path is short when (x, y) is near one component of $\partial\overline{\Upsilon}$ and long near the other. Let u_t be the bend associated to x_t. Given an orientation on $u_0 = u$, we extend it continuously to an orientation on $u_1 = v$. Let $\overrightarrow{u}$ be vector parallel to our oriented u. That is, $\overrightarrow{u}$ points from the tail of u to the head of u. Likewise define $\overrightarrow{v}$. We write $\overrightarrow{u} \rightsquigarrow \overrightarrow{v}$. Since we are on a Moebius band, $\overrightarrow{v} \rightsquigarrow -\overrightarrow{u}$.

The Functions: Let m_u and m_v be the midpoints of u and v. Define

$$g(x, y) = \overrightarrow{u} \cdot \overrightarrow{v}, \qquad h(x, y) = (m_u - m_v) \cdot (\overrightarrow{u} \times \overrightarrow{v}). \tag{4}$$

If we had started with the other orientation of u we would get the same value for g and h because $-\overrightarrow{u} \rightsquigarrow -\overrightarrow{v}$. Hence g and h are well defined. Note that g and h extend continuously to $\overline{\Upsilon}$. We have $g \geq 1$ on one component of $\partial\overline{\Upsilon}$ and $g \leq -1$ on the other. We have $h = 0$ on $\partial\overline{\Upsilon}$. We compute

$$g(y, x) = \overrightarrow{v} \cdot (-\overrightarrow{u}) = -g(x, y). \tag{5}$$

$$h(y, x) = (m_v - m_u) \cdot (\overrightarrow{v} \times (-\overrightarrow{u})) = (m_v - m_u) \cdot (\overrightarrow{u} \times \overrightarrow{v}) = -h(x, y). \tag{6}$$

The map $\Sigma(x, y) = (y, x)$ extends continuously to $\overline{\Upsilon}$ and swaps the boundary components. Equations 5 and 6 say that $g \circ \Sigma = -g$ and $h \circ \Sigma = -h$.

Lemma 3.1 *If $g(x,y) = h(x,y) = 0$ then (u,v) make a T-pattern.*

Proof: Since $g(x,y) = 0$ the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ are orthogonal. Hence $\overrightarrow{n} = \overrightarrow{u} \times \overrightarrow{v}$ is nonzero. By construction u and v and the segment $\overline{m_u m_v}$ all lie in planes orthogonal to $\overrightarrow{n}$. But then they all lie in the same plane orthogonal to $\overrightarrow{n}$. In short, u and v are co-planar. ♠

To prove Lemma T, we just have to prove that g and h simultaneously vanish somewhere in Υ. Suppose not. Since $|g| \geq 1$ on $\partial\overline{\Upsilon}$, we know g and h do not simultaneously vanish on $\overline{\Upsilon}$. Let S^1 be the unit circle. Let $A = (f,g)$ and $B = A/\|A\|$. Then $B : \overline{\Upsilon} \to S^1$ is well-defined and continuous. B maps one component of $\partial\overline{\Upsilon}$ to $(1,0)$ and the other to $(-1,0)$.

Consider any path γ which connects a point in one component of $\partial\overline{\Upsilon}$ to a point in the other. The image $B(\gamma)$, <u>always</u> oriented from $(1,0)$ to $(-1,0)$, winds some half integer $w(\gamma)$ times around the origin. All choices of γ are homotopic to each other relative to $\partial\overline{\Upsilon}$. Thus $w(\gamma)$ is independent of γ.

Our independence result says that $w(\Sigma(\gamma)) = w(\gamma)$. On the other hand, $B \circ \Sigma = -B$. So, as Figure 3 illustrates, when we orient $B(\Sigma(\gamma)) = -B(\gamma)$ from $(1,0)$ to $(-1,0)$, the winding number is $-w(\gamma)$. This contradiction completes the proof of Lemma T.

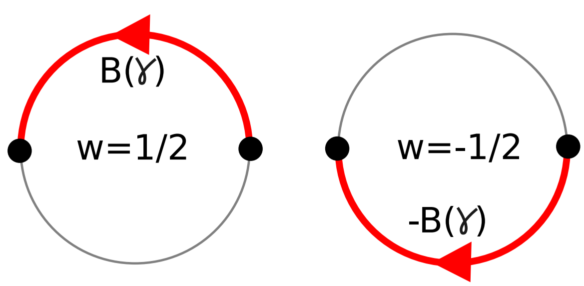

Figure 3: The effect of negation: a cartoon

4 Proof of the Triangular Limit Theorem

We revisit Lemma G. Here is Figure 2.2 again.

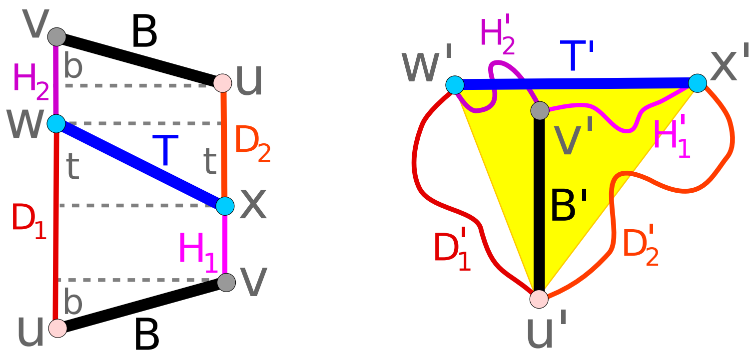

Figure 2.2: The trapezoid (left) and the T-pattern (right).

Here is Equation 2 again.

$$2\lambda = \ell(\partial M_\lambda) = \ell(\partial\Omega) \geq p(\nabla) \geq \sqrt{1+t^2} + \sqrt{5+t^2}. \tag{7}$$

Suppose we have a sequence $\{\Omega_n\}$ of embedded paper Moebius bands with $\lambda_n \to \sqrt{3}$. We run the constructions from Lemma G for each one. Looking at the analysis done at the end of the proof of Lemma G, we see that $t_n \to t_0 = 1/\sqrt{3}$. Also $b_n \to 0$, because otherwise the height of ∇_n, which exceeds $\sqrt{1+b_n^2}$, does not converge to 1. The parameters $b = 0$ and $t = 1/\sqrt{3}$ respectively describe the top/bottom bend B' and the middle bend T' shown on the red strip in Figure 1 (left). We normalize by isometries of M_{λ_n} so that $B'_n \to B'$ and $T'_n \to T'$.

Thanks to the uniqeness in Lemma 2.1, the triangle ∇_n converges up to isometry to the equilateral triangle ∇ of perimeter $2\sqrt{3}$ shown in Figure 1 (right). We normalize by isometries of $\boldsymbol{R}^3$ so that the vertices of ∇_n converge to the vertices of ∇. Inspecting Equation 7, we see that

$$|\ell(\partial\Omega_n) - p(\nabla)| \to 0. \tag{8}$$

Since I_n is length perserving the convergence in Equation 8 implies that I_n, when restricted to each of the 4 segments $D_{n,j}$ and $H_{j,n}$ in ∂M_{λ_n}, converges uniformly to a linear isometry. Hence the restriction of I_n to ∂M_{λ_n} converges uniformly to the map that comes from the triangular Moebius band. The action of I_n on ∂M_{λ_n} determines the action of I_n on M_{λ_n}, so the convergence on the boundary implies the convergence on the whole space. This completes the proof of the Triangular Limit Theorem.

5 Discussion

5.1 Lemma G

The proof of Lemma G only requires the map $I : M_\lambda \to \Omega$ to have the following properties.

1. I is continuous.

2. The interior of M_λ has a continuous partition by open line segments whose endpoints lie in the boundary.

3. Given an arbitrary line segment v in the partition the image $I(v)$ is a line segment in $\boldsymbol{R}^3$ that is at least as long as v.

4. The restriction $I : \partial M_\lambda \to \partial\Omega$ never increases arc-length.

5. There exist 2 segments v, w in the partition such that $I(v)$ and $I(w)$ are disjoint and lie in perpendicular intersecting lines.

The Triangular Limit Theorem does not quite work in this generality, because the restriction of I to ∂M_λ does not determine the action of I on all of M_λ. Nevertheless, we can say that for a minimizing sequence $\{I_n\}$, the maps converge uniformly on the boundary, up to isometry, to the triangular Moebius band map. Also, up to isometries the images Ω_n converge (e.g. in the Hausdorff metric) to the triangular paper Moebius band.

5.2 Lemma T

The proof I give of Lemma T is quite reminiscent of the proof of the Borsuk-Ulam Theorem. Indeed, Jeremy Kahn pointed out to me that the endgame of my proof really is the Borsuk-Ulam proof in disguise. To see this, note that we obtain the 2-sphere S^2 by crushing each component of $\partial\overline{\Upsilon}$ to a point. Then B induces a map $S^2 \to S^1$ with $B \circ \Sigma = -\Sigma$. The map Σ, which is a glide reflection on $\overline{\Upsilon}$, acts on S^2 as the antipodal map.

In this context, it is more natural to redefine the vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ to be the *unit vectors* parallel to the orientations of u and v. Once this is done, the functions g and h themselves restrict to our quotient S^2 and we really have the exact conditions for the Borsuk-Ulam Theorem.

Anton Izosimov and Sergei Tabachnikov independently suggested to me the following general formulation of Lemma T.

Lemma 5.1 *Suppose* $\{L_t|\ t \in [0,1]\}$ *is a continuous family of oriented lines in* $\boldsymbol{R}^3$ *such that* $L_1 = L_0^{\rm opp}$*, the same line as* L_0 *but with the opposite orientation. Then there exist parameters* $r, s \in [0,1]$ *such that* L_r *and* L_s *are perpendicular intersecting lines.*

This result immediately implies Lemma T, and it has essentially the same proof. In particular, Lemma 5.1 applies to maps $I : M_\lambda \to \Omega$ which satisfy Conditions 1-4 above. The output is a T-pattern which might or might not be embedded. If I is an embedding then, of course, the T-pattern will also be embedded.

Sergei also suggested to me a beautiful alternate formalism for the proof of Lemma T. One introduces the *Study numbers*. These have the form $x + \epsilon y$ where $x, y \in \boldsymbol{R}$ and $\epsilon^2 = 0$. Likewise, one introduces the *Study vectors*. These have the form $\overrightarrow{a} + \epsilon \overrightarrow{b}$, where $\overrightarrow{a}, \overrightarrow{b} \in \boldsymbol{R}^3$ and again $\epsilon^2 = 0$. In this context, the dot product of two Study vectors makes sense and is a Study number.

Each oriented line $\ell \subset \boldsymbol{R}^3$ gives rise to a Study vector $\xi_\ell = \overrightarrow{a} + \epsilon \overrightarrow{b}$ where $\overrightarrow{a}$ is the unit vector pointing in the direction of ℓ and $\overrightarrow{b} = \ell' \times \overrightarrow{a}$. Here $\ell' \in \ell$ is any point. All choices of ℓ' give rise to the same $\overrightarrow{b}$; this vector is called the *moment vector* of ℓ. This formalism identifies the space of oriented lines in $\boldsymbol{R}^3$ with the so-called *study sphere* consisting of Study vectors ξ such that $\xi \cdot \xi = 1$. The Study dot product $\xi_\ell \cdot \xi_m$ vanishes if and only if ℓ and m are perpendicular and intersect. Thus our two functions g and h carry the same information as the Study dot product. This makes the functions g and h seem more canonical.

5.3 Folded Ribbon Knots

Elizabeth Denne pointed out to me the connection between paper Moebius bands and *folded ribbon knots*. Her paper with Troy Larsen [**DL**] gives a formal definition of a folded ribbon knot and has a wealth of interesting constructions, results, and conjectures. See also her survey article [**D**].

Informally, folded ribbon knots are the objects you get when you take a flat cylinder or Moebius band, fold it into a knot, and then press it into the plane. Associated to a folded ribbon knot is a polygon, which comes from the centerline of the object. Even though the ribbon knot lies entirely in the plane, one assigns additional combinatorial data which keeps track of "infinitesimal" under and over crossings as in a knot diagram. So the associated centerline is really a knot (or possibly the unknot).

[**DL**, Corollary 25] proves our Main Theorem in the category folded ribbon Moebius bands whose associated polygonal knot is a triangle. This is a finite dimensional problem. [**DL**, Conjecture 26] says that [**DL**, Corollary 25] is true without the restriction that the associated polygonal knot is a triangle, and this is an infinite dimensional problem like the Halpern-Weaver Conjecture.

The combination of our Main Theorem and the Triangular Limit Theorem implies [**DL**, Conjecture 26]. One takes arbitrarily nearby smooth approximations, as in [**HW**], and then applies our results to them. Alteriatively, the same proof that we gave of Lemmas G and T probably would work in this category. (I did not think this through in all details.)

One might also ask about the converse. If it were possible to flatten, through isometric embeddings, an arbitrary paper Moebius band into a knotted ribbon graph, then [**DL**, Conjecture 26] would imply our results. (Again, I did not think this through in all details.) While I do not think that all twisted paper Moebius bands have this property, it might be the case that paper Moebius bands with sufficiently small aspect ratio do have this property. In any case, the possibility of flattening paper Moebius bands isometrically into folded ribbon knots seems like an appealing topic for further investigation.

6 References

[**CF**] Y. Chen and E. Fried, *Mobius bands, unstretchable material sheets and developable surfaces*, Proceedings of the Royal Society A, (2016)

[**CKS**] J. Cantarella, R. Kusner, J. Sullivan, *On the minimum ropelength of knots and links*, Invent. Math. **150** (2) pp 257-286 (2003)

[**D**] E. Denne, *Folded Ribbon Knots in the Plane*, The Encyclopedia of Knot Theory (ed. Colin Adams, Erica Flapan, Allison Henrich, Louis H. Kauffman, Lewis D. Ludwig, Sam Nelson) Chapter 88, CRC Press (2021)

[**DL**] E. Denne, T. Larsen, *Linking number and folded ribbon unknots*, Journal of Knot Theory and Its Ramifications, Vol. 32 No. 1 (2023)

[**DDS**] E. Denne, Y. Diao, J. M. Sullivan, *Quaadrisecants give new lower bounds for the ropelength of a knot*, Geometry&Topology 19 (2006) pp 1–26

[**FT**], D. Fuchs, S. Tabachnikov, *Mathematical Omnibus: Thirty Lectures on Classic Mathematics*, AMS 2007

[**HF**], D.F. Hinz, E. Fried, *Translation of Michael Sadowsky's paper 'An elementary proof for the existence of a developable MÖBIUS band and the attribution of the geometric problem to a variational problem'*. J. Elast. 119, 3–6 (2015)

[**HW**], B. Halpern and C. Weaver, *Inverting a cylinder through isometric immersions and embeddings*, Trans. Am. Math. Soc **230**, pp 41–70 (1977)

[**MK**] L. Mahadevan and J. B. Keller, *The shape of a Möbius band*, Proceedings of the Royal Society A (1993)

[**Sa**], M. Sadowski, *Ein elementarer Beweis für die Existenz eines abwickelbaren MÖBIUSschen Bandes und die Zurückführung des geometrischen Problems auf einVariationsproblem.* Sitzungsberichte der Preussischen Akad. der Wissenschaften, physikalisch-mathematische Klasse 22, 412–415.2 (1930)

[**S**] R. E. Schwartz, *An improved bound on the optimal paper Moebius band*, Geometriae Dedicata, 2021

[**S2**] R. E. Schwartz, *The Optimal Paper Moebius Band: A Friendly Account*, preprint, 2023

[**SH**] E. L. Starostin, G. H. M. van der Heijden, *The shape of a Möbius Strip*, Nature Materials **6** (2007) pp 563 – 567

[**Sz**] G. Schwarz, *A pretender to the title "canonical Moebius strip"*, Pacific J. of Math., **143** (1) pp. 195-200, (1990)

[**T**] Todres, R. E., *Translation of W. Wunderlich's On a Developable Möbius band*, Journal of Elasticity **119** pp 23–34 (2015)

[**W**] W. Wunderlich, *Über ein abwickelbares Möbiusband*, Monatshefte für Mathematik **66** pp 276–289 (1962)