

A short proof that composite twisted unknots are singly twisted unknots

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Abstract

We present a short proof of a theorem of Hayashi and Motegi and (independently) Goodman-Strauss that only singly twisted unknots are composite.

Keywords Knot, unknot, composite knot, twisting, Dehn surgery.

AMS Classification 57M25.

Let K' be an unknotted circle in S^3 , and let D be a disk in S^3 such that $K' \cup \partial D$ is a link L and $|K' \cap D| > 1$ and minimal in the isotopy class of K' in $S^3 - \partial D$. Performing $\frac{1}{n}$ -Dehn surgery on ∂D yields a knot K . We will prove the following

Theorem (Hayashi and Motegi [4], Goodman-Strauss [1]). *If K is composite, then $n = \pm 1$.*

Proof. Let X_L denote the exterior of L . By $X_L(\alpha)$ we shall mean the manifold obtained from X_L by performing α -Dehn surgery on the component of ∂X_L corresponding to ∂D .

We assume the reader is familiar with the machinery surveyed in [2]. $X_L(\infty)$ is a solid torus and so contains a nonseparating disk \widehat{B} . Assuming that K is composite, $X_L(\frac{1}{n})$ contains an essential (non-boundary parallel) separating annulus \widehat{A} whose boundary components are meridians of K . These two surfaces give rise to two labeled graphs G_B and G_A in the usual way [2], and after suitable isotopies we may assume the number of vertices in each minimal and each arc component of $A \cap B$ essential in both A and B . Let a be the number of vertices of G_A , b the number of vertices of G_B . Note that the number of boundary edges in each graph is at most two and that each boundary component of \widehat{A} is incident to a boundary edge—since $\partial \widehat{B}$ is a longitude of K' .

We claim that neither G_A nor G_B represents all types.

Suppose that G_A represents all types. Let \mathcal{D} be a set of (interior) faces of G_A representing all types. Since the edges of elements of \mathcal{D} lie in \widehat{B} , Theorem 4.3 of [2] produces a summand M of S^3 with $H_1 M$ finite and nonzero. So G_A does

not represent all types.

Suppose that G_B represents all types, and let \mathcal{D} be a set of (interior) faces of G_B representing all types. The edges of the elements of \mathcal{D} lie in \widehat{A} , and \widehat{A} lies in a disk in S^3 —just cap off a boundary component with a meridional disk. Again, Theorem 4.3 of [2]—using \widehat{A} capped off rather than \widehat{A} —produces a summand of S^3 with torsion in its first homology group. So G_B does not represent all types.

Note that since a Scharlemann cycle represents all types, neither graph contains a Scharlemann cycle.

Suppose that $|n| \geq 2$. Since neither graph represents all types, G_A contains a $(b-1)$ -web and G_B contains a great a -web, Λ —for G_B , the web is provided by Corollary 3.4 of [3]; For G_A , the web is obtained by interchanging the roles of A and B in the proof of Theorem 3.2 of [3]. Since \widehat{A} is separating, a is even, hence greater than or equal to two. So there is a label x such that for any vertex v in Λ there is an edge in Λ labeled x at v . If $a \geq 4$ or $|n| \geq 3$, there are two such labels, x and y , say.

Suppose $a \geq 4$. If every label occurs at the endpoint (in Λ) of an edge not belonging to Λ , then there is a third label z possessing the property stated for x and y . If some label w does not occur in this fashion, then for every vertex v of Λ there are two edges in Λ labeled w at v . In either case, the fact that there are at most two boundary edges implies that there is a label u such that for every vertex v in Λ , there is an interior edge of Λ labeled u at v . This yields a great u -cycle, hence a Scharlemann cycle, in G_B . So $a = 2$.

Suppose that $|n| \geq 3$. In this case, one of the labels, x say, has the property that every vertex v in Λ is incident to two edges in Λ labeled x at v . Again, this gives rise to a Scharlemann cycle in G_B . So $a = 2$ and $|n| = 2$.

The two vertices v and w of G_A are of opposite sign since \widehat{A} is separating, and so the $(b-1)$ -web in G_A has a single vertex, v say. Since $|n| \geq 2$, there is a loop at v .

If this loop is inessential in \widehat{A} , then it must bound a disk containing w . So, every edge incident to w must be incident to v . Since there are $2b$ edges incident to each vertex, there are $2b$ edges joining w and v . But this contradicts the fact that G_A contains a $(b-1)$ -web. *Note:* It is tempting to use the fact that there are boundary edges in G_A to obtain a contradiction here, but note that we have not bothered to exclude the possibility of a single boundary edge running from one component of $\partial\widehat{A}$ to the other.

So the loop at v separates \widehat{A} into two annuli, A_1 and A_2 . One of these, A_2 say, contains w . Since A_1 contains a component of $\partial\widehat{A}$, A_1 contains a single boundary edge. If the other boundary edge of G_A is incident to w , then G_A must appear

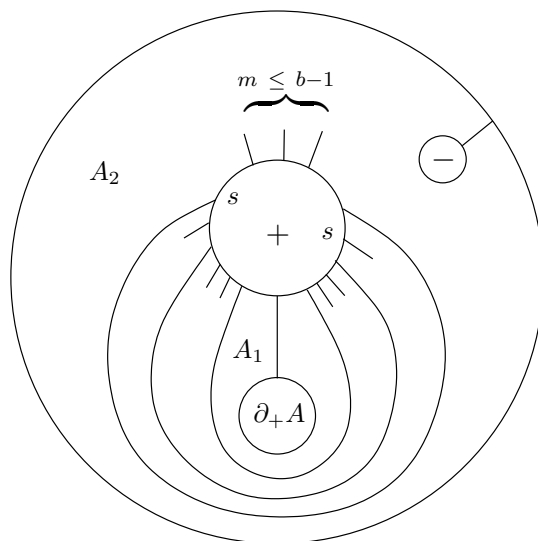


Figure 1: A great s -cycle is forced.

as in Figure 1. But this forces a great s -cycle in G_A , which must contain a Scharlemann cycle, a contradiction. So the other boundary edge is incident to v . But now cutting \hat{A} along the union of v and the boundary edges leaves a disk that contains w and all of its neighboring edges. So there are $2b$ edges joining v and w , again contradicting the fact that G_A contains a $(b-1)$ -web. So $n = \pm 1$. \square

Acknowledgements

The author would like to thank his advisor Cameron Gordon. This work supported in part by TARP grant 003658-0519-2001.

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