Bundles, handcuffs, and local freedom

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Abstract

We answer a question of J. Anderson's by producing infinitely many commensurability classes of fibered hyperbolic 3–manifolds whose fundamental groups contain subgroups that are locally free and not free. These manifolds are obtained by performing 0–surgery on a collection of knots with the same properties.
Keywords: Locally free groups, Fibered knots, Fibered 3-manifolds
Mathematics Subject Classification (2000): 57M25; 57M50, 20E25

> In the case of these *knots* then, and of the several obstructions, which, may it please your reverences, such knots cast in our way in getting through life—every hasty man can whip out his penknife and cut through them.—'Tis wrong. Believe me, Sirs, the most virtuous way, and which both reason and conscience dictate—is to take our teeth or our fingers to them.
>
> Laurence Sterne, From *The Life and Opinions of Tristram Shandy, Gentleman*

1 Introduction

A group is called **locally free** if all of its finitely generated subgroups are free. Of course, free groups have this property, but there are locally free groups that are not free. The additive group of rational numbers is such a group. To obtain a nonabelian example, take a properly ascending union of nonabelian free groups, all of rank less than some fixed bound. The union is locally free, infinitely generated and Hopfian, see [10], 110–111. We will be interested in the following example. Let G be the group defined by the presentation $\mathcal{P} = (a, b, t \mid tat^{-1} = [b, a])$. G admits a surjection to $\mathbb{Z}$ in which the images of b and t are zero and one, respectively. The kernel of this homomorphism is locally free and not free, see [6] and [11], VIII.E.9, for details.

Let P denote the presentation 2–complex associated to $\mathcal{P}$ and consider the embedding of P into S^3 portrayed in Figure 1. Let X be a regular neighborhood of P. The boundary of X is a surface of genus two and, as we shall see below, by carefully choosing another manifold with a genus two boundary component and gluing it to X one can obtain hyperbolic manifolds of finite volume in which X is π_1–injective, see also [1].

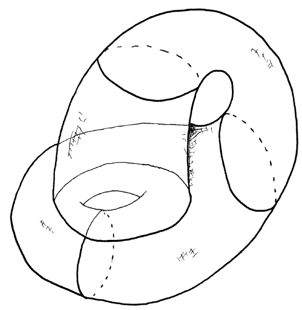

Figure 1: The 2-complex P

Thurston has asked if finite volume hyperbolic 3–manifolds have finite covers that fiber over the circle [17]. So, if the property of having a locally free nonfree subgroup is invariant under taking finite index subgroups, there would be bundles with such subgroups. Now, it is a theorem of Serre that torsion free groups and their finite index subgroups share the same cohomological dimension [12], see also [4], 190-191, and it is a theorem of Stallings [14] and Swan [15] that the only groups of cohomological dimension one are the free ones. So, if a group is locally free and not free, then its finite index subgroups share these properties. With this in mind, J. Anderson [3] has posed the following

Question. *Can the fundamental group of a hyperbolic* 3*–manifold that fibers over the circle contain a subgroup that is locally free and not free?*

By the above remarks, a negative answer would provide hyperbolic 3–manifolds that are not virtually fibered. Our purpose here is to prove the following theorem, which answers this question in the affirmative.

Theorem 1. *There are infinitely many commensurability classes of closed hyperbolic 3–manifolds that fiber over the circle whose fundamental groups contain subgroups that are locally free and not free.*

These manifolds are obtained by performing 0–surgery on the "handcuff" knots pictured in Figure 2, for which we have the

Theorem 2. *The knots pictured in Figure 2 are fibered, hyperbolic, represent infinitely many commensurability classes, and their groups contain subgroups that are locally free and not free.*

The first theorem is obtained by establishing the second and demonstrating that the desired properties survive surgery.

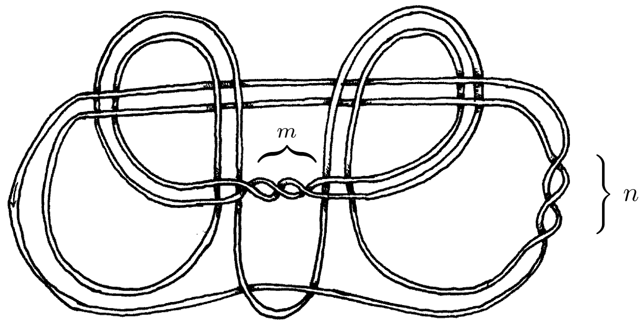

Figure 2: The knot $K_{m,n}$

2 Notation

If G is a group and $g, h \in G$, we denote the normal closure of g in G by $\mathrm{ngp}(g)$, the word $g^{-1}h^{-1}gh$ by $[g, h]$. We let $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ denote the group defined by the presentation $(x_1, \ldots, x_n \mid r_1, \ldots, r_m)$.

We shall suppress any mention of basepoints when considering the fundamental group of a space as we will only be concerned with properties that are invariant under conjugation.

If M is an n–manifold and F is a properly embedded submanifold in M, we write $M \setminus F = M - \mathrm{int\,nhd}(F)$. If F is a bicollared $(n-1)$–manifold, $\mathrm{nhd}(F) \cong [-1, 1] \times F$ and we call the components of $\{-1\} \times F$ and $\{1\} \times F$ the **traces** of F in $M \setminus F$. For two transverse submanifolds F and F', we often write $F \setminus F'$ for $F \setminus (F \cap F')$.

Let M be a 3–manifold with a single torus boundary component T. A **slope** is an isotopy class of an essential embedded circle in T. If α is a slope, we write $M(\alpha)$ to denote the manifold $M \cup V_\alpha$ where V_α is a solid torus glued to M via a homeomorphism $T \to \partial V_\alpha$ taking an element of α to the boundary of a meridian disk in V_α. If α and β are slopes, we let $\Delta(\alpha, \beta)$ denote the minimal geometric intersection number of elements of α and β.

A 3–manifold is said to be **atoroidal** if every one of its embedded incompressible tori is boundary parallel. A properly embedded annulus in a 3–manifold M is **essential** if it is incompressible and is not isotopic rel ∂ to an annulus in ∂M.

Given a topological space X, let $|X|$ denote the number of path components of X.

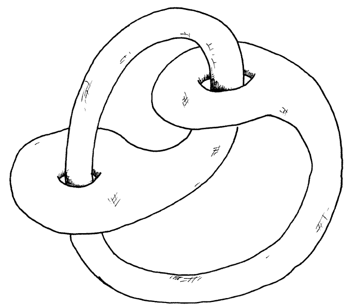

Figure 3: The complement of int X

3 The Construction

Let X be as in the introduction. Note that since $X \subset S^3$ and ∂X is connected, X is irreducible.

$F = \partial X$ is incompressible in X as follows. F cannot compress completely for X would then be a handlebody—since X is irreducible—and $\pi_1(X) \cong G$ contains a nonfree subgroup. So, if F were compressible, X would contain an incompressible torus—since F is a surface of genus two. In particular, $G \cong \pi_1(X)$ would contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Now, in [11], VIII.E.9, Maskit constructs a purely loxodromic discrete faithful representation of G into $\mathrm{PSL}(2, \mathbb{C})$. This implies that G contains no subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. So F is incompressible. Note that int X need not be homeomorphic to the quotient of $\mathbb{H}^3$ by the image of Maskit's representation.

Up to an isotopy of S^3, $S^3 - \text{int } X$ is the handlebody H of Figure 3. Fix m and n odd (and positive), and let $K = K_{m,n}$ be the simple closed curve in H shown in Figure 4, $Y = H \setminus K$. We fix throughout a homeomorphism between the handlebody of Figure 4 and that of Figure 3 taking the curve K to the knot shown in Figure 2.

We claim that F is incompressible in Y. To see this, note that if there were a compressing disk D, $H \setminus D$ would be a disjoint union of one or two solid tori. Supposing this to be the case, the core of each solid torus represents a primitive element in $\pi_1(H)$. Since $D \subset Y$, K is contained in one of these solid tori and so $[K]$ must lie in a nontrivial free factor of $\pi_1(H)$. So, $\pi_1(H)/\mathrm{ngp}([K])$ is either a nontrivial free product or infinite cyclic, neither of which occurs since $\pi_1(H)/\mathrm{ngp}([K])$ is the Klein bottle group $\langle x, y \mid x^2 y^2 \rangle$. So F is incompressible in Y.

Note also that Y is irreducible, for if S is a sphere in Y, it must bound a ball in H, and since $[K] \neq 1$ in $\pi_1(H)$, K lies outside of this ball, and so S bounds a ball in Y.

Let $M = M_{m,n} = X \cup_F Y = S^3 \setminus K$. Since F is incompressible in X and Y, $\pi_1(M) \cong \pi_1(X) *_{\pi_1(F)} \pi_1(Y)$ and so $G \cong \pi_1(X)$ embeds in $\pi_1(M)$. In particular,

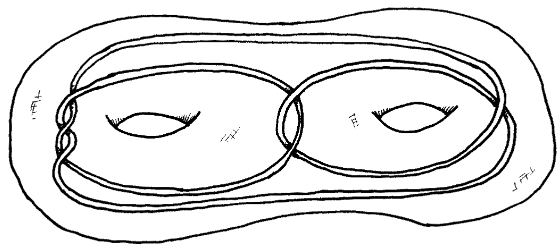

Figure 4: The knot inside the handlebody

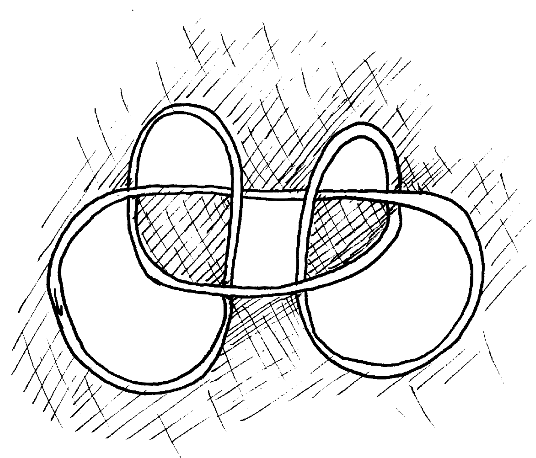

Figure 5: Square knot and fiber

$\pi_1(M)$ contains a subgroup that is locally free and not free.

□

4 Proof that K is fibered

We reconstruct K using the techniques of [13] in order to see the fibration.

We begin by considering the square knot L_1 with fiber of Euler characteristic -3 as shown in Figure 5.

We next form the $(2, m+n)$ cable of L_1 to obtain a link L_2, Figure 6. L_2 is again fibered, and we shall need to identify a fiber. The cable is fibered in a solid torus as pictured in Figure 7. The Euler characteristic of this fiber is $-(m+n)$. Since the boundary of L_1's fiber is the longitude given by the blackboard framing, we may assume that L_2 appears as in Figure 6 and that its

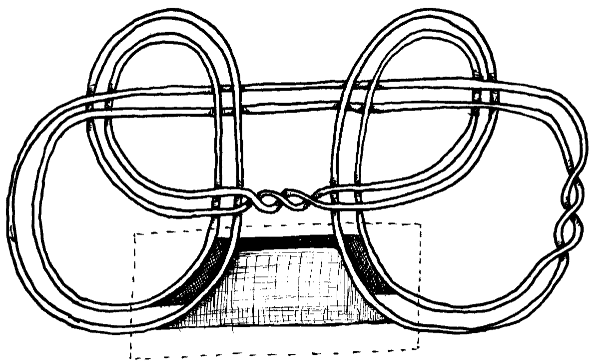

Figure 6: Cabled square knot and patch of fiber

fiber—consisting of two copies of L_1's fiber and one from the cable—is in the position indicated in the highlighted region.

Plumbing on the Hopf band as shown in Figure 8 yields a new fiber and the knot K. The fiber has Euler characteristic $2(-3)-(m+n)-1 = -(6+m+n)-1$ and so its genus is $4 + \frac{m+n}{2}$. So, the genera of the fibers go to infinity with m and n, and the collection of knots is infinite.

□

5 Proof that $\operatorname{int} M$ admits a hyperbolic structure

By Thurston's uniformization theorem for Haken manifolds [17, 9], to show that K is hyperbolic, we need only demonstrate that M is irreducible, atoroidal, and not Seifert fibered. M is a knot exterior and so irreducible.

If M admitted a Seifert fibering, then F (as in Section 3) would be isotopic to a horizontal surface. If this were the case, $M \setminus F$ would be an I–bundle over some surface. In particular, $\pi_1(X)$ would be a surface group. But every locally free subgroup of a surface group is free. So M is not a Seifert fibered space.

The proof that M is atoroidal proceeds as follows. Let T' be an incompressible torus in M. We may take $T' \cap F \subset T'$ to be a (possibly empty) collection of simple closed curves. If any of these is inessential, then there is an innermost such, say γ. The disk γ bounds, D say, is contained in X or Y and since F is incompressible in both, γ must bound a disk D' in F. Since M is irreducible, $D \cup D'$ bounds a ball B. We use B to isotope T' to a torus that intersects F in fewer simple closed curves. Continuing in this manner we obtain a torus T'' such that $T'' \cap F \subset T''$ is a (possibly empty) collection of essential curves, which must all be parallel. So, if $T'' \cap F$ is nonempty, $T'' \cap Y$ and $T'' \cap X$ consist entirely of annuli. If any annulus in $T'' \cap X$ is boundary parallel, we

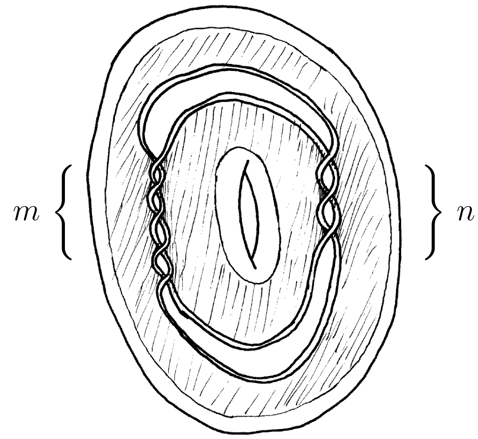

Figure 7: Fibering the cable in a solid torus

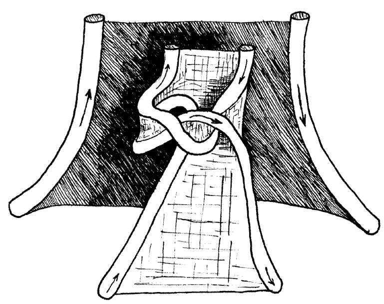

Figure 8: Plumbing

may isotope T'' to reduce $|T'' \cap F|$. So we may assume that every annulus in $T'' \cap X$ is essential in X. We shall need the following

Lemma. *The three annuli pictured in Figure 9 are the only essential annuli in X (up to ambient isotopy).*

We postpone the proof of the lemma and continue the proof of Theorem 2.

As in Section 3, let $H = S^3 - \text{int } X$. In $F = \partial H$, each component of ∂A_i is isotopic to one of the three curves in Figure 10. By the lemma, each component of $T'' \cap X$ is one of the A_i, and as each of the A_i has an α_j as a boundary component, some α_k must be a boundary component of some component C of $T'' \cap Y$. Since α_1, α_2 and β are pairwise non–homologous in H, the other

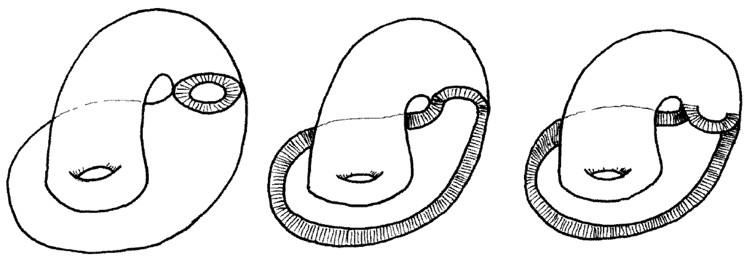

Figure 9: The three annuli

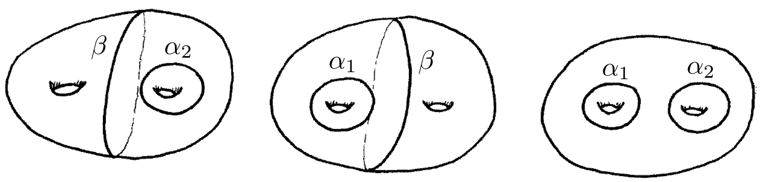

Figure 10: The boundary components of the three annuli

boundary component of C is parallel to α_k. So there is an annulus C' in ∂H with $\partial C' = \partial C$. Now, any torus in a handlebody either bounds a solid torus or is interior to a ball. Since α_k is primitive in $\pi_1(H)$, $C \cup C' \subset H$ cannot be interior to a ball. So $C \cup C'$ bounds a solid torus $E \subset H$. Note also that $[\alpha_k]$ being primitive in $\pi_1(H)$ implies that $[\alpha_k]$ is primitive in $\pi_1(E)$—otherwise $[\alpha_k]$ would be a proper power of the core of E. If K were contained in E, then $[K]$ would lie in $\langle[\alpha_k]\rangle$, a proper free factor of $\pi_1(H)$, contrary to the choice of K. So, K is not contained in E and since $[\alpha_k]$ is primitive in $\pi_1(E)$, C is isotopic rel ∂ into X. So we may isotope T'' to a torus T''' with $|T''' \cap F| < |T'' \cap F|$. Again we may arrange—by another isotopy—that every annulus in $T''' \cap X$ is essential in X. We continue this procedure until we have a torus T such that every component of $T \cap X$ is essential in X and no component of $T \cap F$ is isotopic to α_1 or α_2. But the nature of the A_i demands that if this occurs, then we have $T \cap F = \emptyset$. Since X is atoroidal, $T \subset Y$.

Let $\widehat{D}_1$ and $\widehat{D}_2$ be the two meridional disks in H pictured in Figure 11, $D_i = \widehat{D}_i \cap Y$ for $i \in \{1, 2\}$. Let $Z = Y \setminus (D_1 \cup D_2)$. By an innermost disk argument employing the irreducibility of Y, we may assume that $T \cap Z$ is a collection of incompressible annuli. Let A be such an annulus. Each component

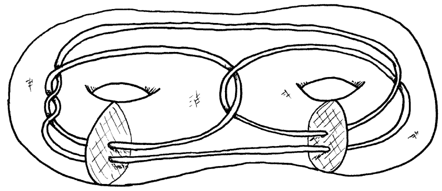

Figure 11: Knot, handlebody, and meridional disks

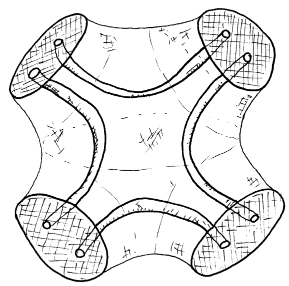

Figure 12: The handlebody Z

of ∂A is contained in one of the traces of the D_i. It is easy to see that a pair of simple closed curves contained in the traces of the D_i are homologous in Z if and only if for some $j \in \{1, 2, 3, 4\}$, they are both isotopic to a boundary component of a_j, where the a_j are the annuli pictured in Figure 12. If ∂A is contained in a single trace of the D_i, then ∂A is the boundary of an annulus A' contained in that trace. As in the argument of the previous paragraph concerning $C \cup C'$, $A \cup A'$ bounds a solid torus in Z that we may use to isotope A to A' relative to its boundary. This yields an isotopy of T that reduces $|T \cap (D_1 \cup D_2)|$. So we may assume that either $T \cap (D_1 \cup D_2) = \emptyset$, in which case $T \subset Z$, or that for each component A of $T \cap Z$, the two components of ∂A lie in different traces of the D_i. The first case is impossible since T is incompressible and Z is a handlebody. So, since the cores of the a_i are primitive in $\pi_1(Z)$, A is parallel to a_j and we conclude that T is boundary parallel.

So, M is atoroidal. Note that the above includes a proof that Y is atoroidal. We will need this fact in Section 6.

To obtain infinitely many commensurability classes, we simply observe that all of the $M_{m,n}$ can be obtained by performing surgery on the link L pictured in Figure 13 and so

$$\mathrm{Vol}(M_{m,n}) \leq v_3 \; || \; [S^3 \setminus L, \partial(S^3 \setminus L)] \; ||$$

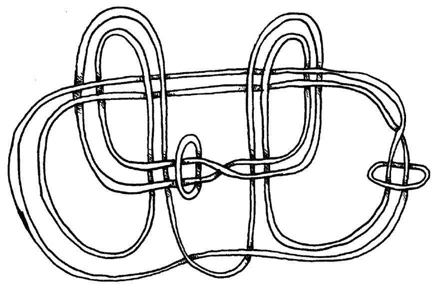

Figure 13: The link L

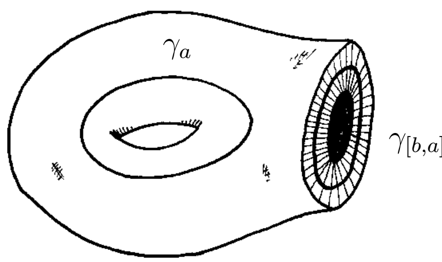

Figure 14: The curves γ_a and $\gamma_{[b,a]}$

where $|| \, [\, \cdot \, , \partial \, \cdot \,] \, ||$ denotes the relative Gromov norm, and v_3 is the volume of a regular ideal 3–simplex [16]. Theorem 0.1 of [2] states that any such collection of hyperbolic 3–manifolds contains representatives from infinitely many commensurability classes. This completes the proof of Theorem 2 modulo the lemma.

As the proof of the lemma is somewhat involved, we draw an outline before proceeding.

Note that $W = X \setminus A_1$ is homeomorphic to $P \times [-1, 1]$ where P is a punctured torus. We choose generators a and b for $\pi_1(W)$ so that the curves $\gamma_a \subset P \times \{1\}$ and $\gamma_{[b,a]} = \partial P \times \{0\}$ pictured in Figure 14 represent a and $[b, a]$ respectively. Note that the traces of A_1 are regular neighborhoods of γ_a and $\gamma_{[b,a]}$, see Figures 14 and 9. Note also that $\partial W \setminus (\gamma_a \cup \gamma_{[b,a]})$ is the disjoint union of a pair of pants $P_+ = (P \times \{1\}) \setminus \gamma_a$ and a punctured torus $P_- = P \times \{-1\}$, see Figure 14.

We consider an arbitrary essential annulus $A \subset X$. The first step of the proof is concerned with constructing an isotopy (rel ∂) that pushes A off of A_1. An innermost disk argument allows us to assume that $A \cap A_1$ is a collection of arcs and essential closed curves. An outermost disk argument eliminates all arcs of intersection that are boundary parallel in A. The fact that the cores of the traces of A_1 in ∂W are γ_a and $\gamma_{[b,a]}$ and the complement of these curves in ∂W is of a special sort allows us to eliminate essential arcs and essential closed curves.

Once A has been isotoped clear of A_1, we consider it as an annulus in W and show that its boundary components lie in opposite ends of the product structure on W. This implies that A is isotopic to a vertical annulus that misses γ_a and the only such annuli are pictured in Figure 9.

Proof of the lemma. First of all note that the A_i are indeed essential as all three are nonseparating. In the following we will assume that A and A_1 are transverse and that the intersection of A with $\mathrm{nhd}(A_1) \cong [-1,1] \times A_1$ is of the form $[-1,1] \times (A \cap A_1)$.

We begin by showing that any essential annulus A can be taken to miss A_1 entirely. If $A \cap A_1$ is empty, we are done. If not, then an innermost disk argument shows that $A \cap A_1 \subset A$ can be taken to be a collection of arcs and essential simple closed curves. Suppose there is a boundary parallel arc component and let D be an outermost disk in $A \setminus A_1$. Let τ denote the arc of ∂D interior to A. If τ is boundary parallel in A_1, it is part of the boundary of a disk $D' \subset A_1$ and $\partial(D \cup D')$ lies in F and so bounds a disk D'' in F. So, $D \cup D' \cup D''$ bounds a ball that we may use to isotope D across A_1. If τ is an essential arc in A_1, D would allow a boundary compression of A_1, leaving a disk B. Now, B is boundary parallel and so A_1 must be separating. But A_1 is nonseparating. So all arc components may be taken to run from one component of ∂A to the other.

Suppose that $A \cap A_1$ contains such an arc. Consider the traces $\{-1\} \times A_1$ and $\{1\} \times A_1$ in $W = X \setminus A_1$, which are regular neighborhoods of γ_a and $\gamma_{[b,a]}$ respectively, say. $A \setminus A_1$ is a collection of disks in the handlebody W that we will consider as 2–dimensional 1–handles $h = [-1,1] \times [-1,1]$, where $\partial_- h = \{-1\} \times [-1,1]$ and $\partial_+ h = \{1\} \times [-1,1]$ lie in the traces of A_1. Let $\delta_- h = [-1,1] \times \{-1\}$ and $\delta_+ h = [-1,1] \times \{1\}$.

If such a 1–handle h has $\partial_- h$ and $\partial_+ h$ lying in different traces, then h is a properly embedded disk in W that intersects $\gamma_{[b,a]}$ in a single point, which is impossible since $\gamma_{[b,a]}$ is separating in ∂W. So each 1–handle h in $A \setminus A_1$ has both $\partial_- h$ and $\partial_+ h$ lying in a single trace of A_1. Since $A \cap \mathrm{nhd}(A_1) = [-1,1] \times (A \cap A_1)$, some 1–handle $h_{[b,a]}$ in this collection must intersect $\gamma_{[b,a]}$ and some other 1–handle h_a must intersect γ_a.

Suppose that $\delta_- h_{[b,a]}$, say, lies in the pair of pants P_+.

If $\delta_- h_{[b,a]}$ separates the two traces of γ_a in P_+, then $\delta_- h_a$ and $\delta_+ h_a$ are both isotopic into γ_a. This implies that ∂h_a either bounds a disk D in ∂W or is isotopic to γ_a. The latter case is impossible since a is primitive in $\pi_1(W)$. In the former, we may use the ball bounded by $h_a \cup D$ to isotope A to decrease $|A \cap A_1|$ by two.

If $\delta_- h_{[b,a]}$ does not separate the two traces of γ_a in the pair of pants, then it is isotopic into $\gamma_{[b,a]}$. Therefore $\partial h_{[b,a]}$ is isotopic into the punctured torus P_-.

Now, P_- is π_1–injective in W and so $\partial h_{[b,a]}$ bounds a disk $D \subset P_-$. Just as above, $h_{[b,a]} \cup D$ bounds a ball that we may use to isotope A in order to decrease $|A \cap A_1|$ by two.

If $\delta_- h_{[b,a]}$ and $\delta_+ h_{[b,a]}$ both lie in P_-, then $\partial h_{[b,a]}$ is isotopic into P_-, and so bounds a disk in ∂W as before and we may isotope A to decrease $|A \cap A_1|$.

Continuing in this fashion, we may assume that there are no arcs in $A \cap A_1$.

Suppose there is an essential closed curve in $A \cap A_1 \subset A$. If there is more than one such curve, we may choose an innermost annulus A' in $A \setminus A_1$. This annulus cobounds a solid torus E in W with some annulus A'' in ∂W—since $A' \cup A''$ is not interior to a ball in W. The core of A'' is freely homotopic to γ_a or $\gamma_{[b,a]}$ and since neither of these represents a proper power in $\pi_1(W)$, the core of A''—hence the core of A'—is primitive in $\pi_1(E)$. This allows us to isotope A' across E to A'' and reduce $|A \cap A_1|$. Continuing this process, we may assume that $A \cap A_1$ is empty or a single curve.

Suppose that we are in the latter case. If a component of ∂A lies in the pair of pants P_+, then it is isotopic in ∂W to γ_a or $\gamma_{[b,a]}$. This yields an isotopy in ∂X of this boundary component across a component of ∂A_1. This increases $|A \cap A_1|$ to two and the above argument allows us to isotope A off of A_1. So both boundary components may be assumed to lie in the punctured torus. If one of these is isotopic in ∂W to $\gamma_{[b,a]}$, then we may again increase $|A \cap A_1|$ to two. If neither of the two curves is isotopic to $\gamma_{[b,a]}$, then they are parallel in P_+, hence parallel in $\partial X - \partial A_1$. Note that this is true of any two disjoint non–boundary parallel simple closed curves in a punctured torus—simply cut along one of the curves to obtain a pair of pants. Since any torus in X is either interior to a ball or bounds a solid torus, and $[A \cap A_1] \in \pi_1(X)$ is conjugate to $a \neq 1$, A cobounds a solid torus $E \subset X$ with some annulus $A' \subset \partial X - \partial A_1$. Since $|A \cap A_1| = 1$, one boundary component of A_1 lies in A'. This contradicts the fact that $A' \subset \partial X - \partial A_1$.

So we may assume that $A \cap A_1 = \emptyset$. In particular, $\partial A \subset P_+ \cup P_-$.

Note that any annulus in ∂W whose boundary misses the traces of A_1 and is essential in X is isotopic to one of the traces of A_1 and is hence isotopic to A_1 in X. So we may assume that our annulus is non–boundary parallel in W.

We now claim that the two components of ∂A lie in opposite ends of $W = P \times [-1, 1]$.

Suppose that ∂A lies in the pair of pants P_+. Then both components are parallel in P_+ and so A cobounds a solid torus E with some annulus $A' \subset P_+$. Every simple closed curve in P_+ is isotopic to γ_a or $\gamma_{[b,a]}$ in ∂W and since neither of these is a proper power in $\pi_1(W)$, the core of A is primitive in $\pi_1(E)$ and so A is boundary parallel in X.

Suppose that ∂A lies in the punctured torus P_-. Note that any essential simple closed curve in P_- is either primitive in $\pi_1(P_-)$ or boundary parallel. If neither component of ∂A is parallel to $\gamma_{[b,a]}$, then the two components are parallel to each other. If one component is parallel to $\gamma_{[b,a]}$, then they both are—since $[b, a]$ is not primitive. In either case, A cobounds a solid torus E

with an annulus $A' \subset P_-$. The core of A' does not represent a proper power in $\pi_1(P_-) \cong \pi_1(W)$ and so represents a primitive element of $\pi_1(E)$. So A is parallel to $A' \subset \partial X$.

We conclude that the two components of ∂A lie in opposite ends of $P \times [-1, 1]$. Such an annulus is isotopic to one that is vertical. Since one component of ∂A lies in the pair of pants $P_+ = (P \times \{1\}) \setminus \gamma_a$, A must be one of $\gamma_{[b,a]} \times [-1, 1]$, $\gamma_a^+ \times [-1, 1]$, or $\gamma_a^- \times [-1, 1]$, where γ_a^+ and γ_a^- denote the traces of γ_a in P_+. These are the annuli A_1, A_2, and A_3, respectively.

This completes the proof of the lemma and so the proof of Theorem 2.

□

6 Closed examples

Proof of Theorem 1. Fix m and n odd and let $K = K_{m,n}$ be the knot in Figure 2. As before, let $M = M_{m,n}$ denote the exterior of K in S^3. We will show that performing 0–surgery on M yields a manifold that is irreducible, atoroidal, and not Seifert fibered. An application of Thurston's uniformization theorem demonstrates that these manifolds admit a hyperbolic structure. Of course, these manifolds are again fibered and the genera of the fibers tell us that the collection $\{M_{m,n}(0)\}$ is infinite.

There is an annulus A running from the surface F to ∂M. To see this, consider the curve K in the handlebody of Figure 4. If we consider the boundary of this handlebody as the standard Heegaard surface for S^3, then K is a pretzel knot and there is an isotopy (inside the handlebody) carrying K into the boundary. The curve obtained by this isotopy is pictured in Figure 15. Inspection reveals that the coannular slope α is $m + n + 2$ relative to the blackboard framing of Figure 2, see Figure 15. Inspection again reveals that the longitude of K has slope $\gamma = -(m + n)$ relative to this framing, Figure 16. So $\Delta = \Delta(\alpha, \gamma) = 2(m + n + 1) > 1$ and so F remains incompressible in $M(\gamma)$ by Theorem 2.4.3 of [5]. In particular, G injects into $\pi_1(M(\gamma))$.

Now consider $Y(\gamma)$. By the above, $F = \partial Y(\gamma)$ is incompressible in $Y(\gamma)$. Note that F compresses in $Y(\alpha)$. In fact, it is not difficult to see that $Y(\alpha)$ is a boundary sum of a solid torus and a twisted I–bundle over the Klein bottle. In particular, $Y(\alpha)$ admits no hyperbolic structure. Since Y is atoroidal and $\Delta = 2(m + n + 1) \geq 6$, Theorem 1.3 of [7] implies that int $Y(\gamma)$ admits a hyperbolic structure. In particular, $Y(\gamma)$ is irreducible and atoroidal. Note that $M(\gamma)$ is hence irreducible. Also note that since F is incompressible in $M(\gamma)$, the argument of Section 5 demonstrating that M does not admit a Seifert fibering applies to show that $M(\gamma)$ does not either.

Suppose that $M(\gamma)$ contains an incompressible torus T. Now, just as in the proof of Theorem 2, we may assume that $T \cap X$ and $T \cap Y(\gamma)$ are collections of annuli. We may also assume, as in the proof of Theorem 2, that the annuli in $T \cap X$ are essential. Again, every boundary component of such an annulus is one of the curves α_1, α_2, or β in F pictured in Figure 10. Let $\widehat{P}$ be a component of $T \cap Y(\gamma)$. Since the boundary component of A that lies in F necessarily

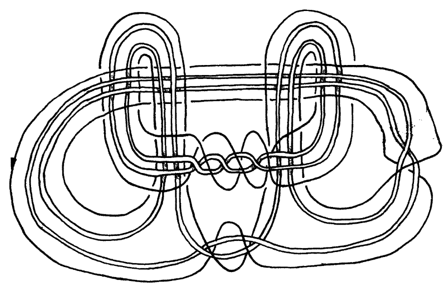

Figure 15: A parallel pushoff of the coannular slope α

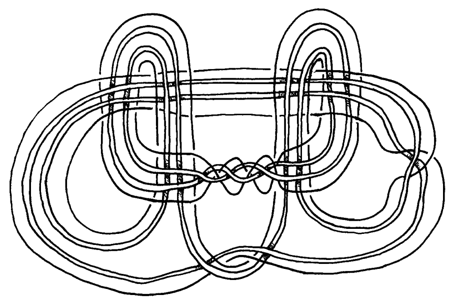

Figure 16: A parallel pushoff of the 0–slope β

intersects α_1, α_2, and β, $\widehat{P} \cap A$ is necessarily nonempty. Furthermore, we may assume that $\widehat{P}$ intersects the core of the filling torus V_γ, for if not, $\widehat{P}$ may be isotoped into Y and the arguments in the proof of Theorem 2 show that we may isotope $\widehat{P}$ into X. We may isotope $\widehat{P}$ so that $\widehat{P} \cap V_\gamma \subset V_\gamma$ is a collection of meridional disks. Let $P = \widehat{P} \cap Y$ and let $\partial_+ Y$ denote the torus component of ∂Y. The components of ∂P lying in $\partial_+ Y$ have slope γ. An isotopy of P now ensures that each boundary component of P lying in $\partial_+ Y$ intersects the boundary of A in exactly Δ points. Yet another isotopy of P ensures that each arc component of $P \cap A$ is essential in both P and A.

The intersection of P and A gives rise to two labeled graphs G_A and G_P in the usual way, see [8], where we consider the component of ∂A lying in the torus boundary component of Y as the only vertex in G_A. Since $\Delta > 2$, the valence of every vertex in G_P is at least three, and so for some vertex v in G_P and some component S of $\partial P \cap F$, there must be a pair of edges in G_P each having an endpoint in v and the other in S. So, there is a disk D_1 in G_P incident to v with the property that (int D_1) $\cap A = \emptyset$, see Figure 17. Without loss of generality, we assume that the label of v is **1**. The two arcs of ∂D_1 interior to P

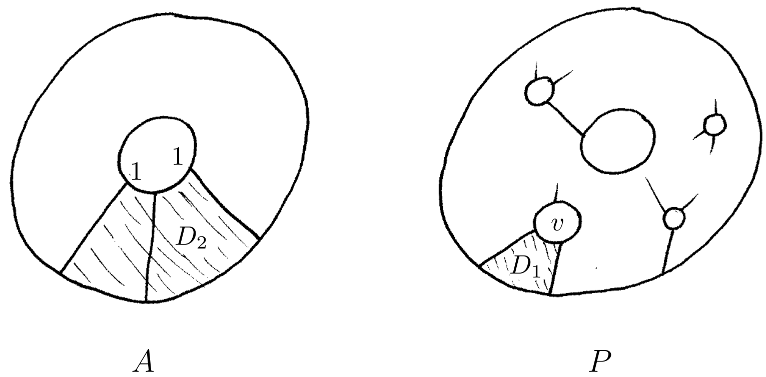

Figure 17: The graphs G_A and G_P

are also edges in G_A, where they are part of the boundary of a disk D_2, Figure 17. Note that $Q = D_1 \cup D_2$ is an annulus. The boundary component of Q lying in $\partial_+ Y$, call it $\partial_+ Q$, is pictured (after a small isotopy) in Figure 18, and apparently intersects α in a single point. So $A \cap Q$ is a single arc. Let $\partial_- A$ and $\partial_- Q$ be the boundary components of these annuli lying in F. These two curves intersect in a single point. Since F is incompressible in Y, the commutator of these curves is nontrivial in $\pi_1(Y)$. But, $[\alpha, \partial_+ Q]$ is trivial in $\pi_1(Y)$ and we may use the annuli A and Q to freely homotope $[\partial_- A, \partial_- Q]$ to $[\alpha, \partial_+ Q]$, which is impossible. We must conclude that P does not exist and so neither does T. So $M(\gamma)$ is atoroidal.

Again,

$$\mathrm{Vol}(M_{m,n}(\gamma)) \leq v_3 \; || \; [S^3 \setminus L, \partial(S^3 \setminus L)] \; ||$$

and so we obtain infinitely many commensurability classes as in Theorem 2. This completes the proof of Theorem 1.

□

7 Acknowledgements

The author wishes to thank Alan Reid and his advisor Cameron Gordon for useful conversations, the latter especially for an observation that simplified the proof of the lemma considerably. This work supported in part by TARP grant 003658-0519-2001.

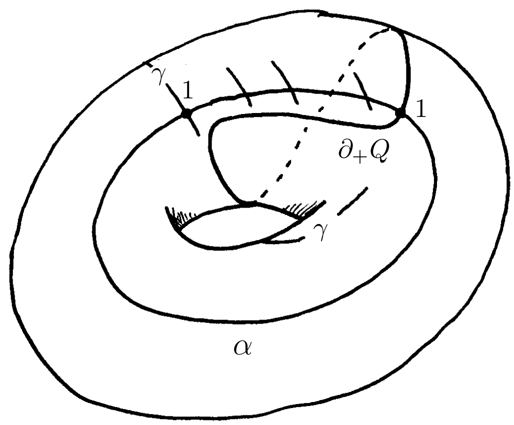

Figure 18: The boundary of Q

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