Achievable ranks of intersections of finitely generated free groups

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Abstract

We answer a question due to A. Myasnikov by proving that all expected ranks occur as the ranks of intersections of finitely generated subgroups of free groups.

Let F be a free group. Let H and K be nontrivial finitely generated subgroups of F. It is a theorem of Howson [1] that $H \cap K$ has finite rank. H. Neumann proved in [2] that $\text{rank}(H \cap K) - 1 \leq 2(\text{rank}(H) - 1)(\text{rank}(K) - 1)$ and asked whether or not $\text{rank}(H \cap K) - 1 \leq (\text{rank}(H) - 1)(\text{rank}(K) - 1)$.

A. Myasnikov has asked which values between 1 and $(m-1)(n-1)$ can be achieved as $\text{rank}(H \cap K) - 1$ for subgroups H and K of ranks m and n—this is problem AUX1 of [4]. We prove that all such numbers occur by proving the following

Theorem. *Let $F(a,b)$ be a free group of rank two. Let*

$$\begin{aligned} H^m_{k,\ell} = \langle a,\ bab^{-1}, \ldots,\ b^k a b^{-k},\ b^{k+1} a^{n-\ell} b^{-(k+1)}, \\ b^{k+2} a^n b^{-(k+2)},\ b^{k+3} a^n b^{-(k+3)}, \ldots,\ b^{m-1} a^n b^{1-m} \rangle \end{aligned}$$

and let $K = \langle b,\ aba^{-1}, \ldots,\ a^{n-1} b a^{1-n} \rangle$, where $0 \leq k \leq m-2$ and $0 \leq \ell \leq n-1$. Then the rank of $H^m_{k,\ell} \cap K$ is $k(n-1) + \ell$.

Corollary. *Let F be a free group and let $m, n \geq 2$ be natural numbers. Let N be a natural number such that $1 \leq N - 1 \leq (m-1)(n-1)$. Then there exist subgroups $H, K \leq F$, of ranks m and n, such that the rank of $H \cap K$ is N.*

Proof of the corollary. The theorem produces the desired subgroups for all N with $N - 1 \leq (m-1)(n-1) - 1$ after passing to a rank two subgroup of F. For $N - 1 = (m-1)(n-1)$, simply let $H = \langle a,\ bab^{-1}, \ldots,\ b^{m-2} a b^{2-m},\ b^{m-1} \rangle$ and let $K = \langle b,\ aba^{-1}, \ldots,\ a^{n-2} b a^{2-n},\ a^{n-1} \rangle$. □

Proof of the theorem. Let X be a wedge of two circles and base $\pi_1(X)$ at the wedge point. We identify $\pi_1(X)$ with $F = F(a,b)$ by calling the homotopy class

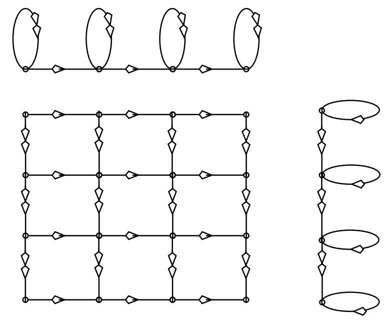

Figure 1: H, K, and $H \cap K$ when $k = m-2$, $\ell = n-1$

of one oriented circle a and the other b. Given a finitely generated subgroup of F, there is a covering space $\widetilde{X}$ corresponding to this subgroup. Moreover, there is a compact subgraph of $\widetilde{X}$ that carries the given subgroup. Given two subgroups and their associated finite graphs, one may construct the graph associated to their intersection. These procedures are laid out carefully in [3] and we assume that the reader is familiar with that paper.

In the figures, the graph associated to H appears at the top, that of K to the right, and that of $H \cap K$ in the center. Edges labelled with two arrowheads represent a, those with one arrowhead represent b. Our basepoint in the graph associated to $H \cap K$ is always the vertex in the upperlefthand corner.

For the moment, fix $k = m-2$. In Figure 1, $\ell = n-1$ and the rank of $H^m_{m-2,n-1} \cap K$ is visibly $(m-1)(n-1)$. Decreasing ℓ by one alters the intersection graph as depicted in Figure 2 and the rank of $H^m_{m-2,n-2} \cap K$ is $(m-1)(n-1)-1$. Figure 3 shows the case when $\ell = n-3$ and the rank of the intersection is $(m-1)(n-1)-2$. When $\ell = n-j$, the rank of $H^m_{m-2,n-j} \cap K$ is $(m-1)(n-1)-(j-1)$.

Figure 4 depicts the case $\ell = 0$. Note that the graph associated to $H^m_{m-2,0} \cap K$ is the graph associated to $H^{m-1}_{m-3,n-1} \cap K$ to which a collection of trees have been attached at their roots, the graph associated to $H^m_{m-3,n-2} \cap K$ is the graph associated to $H^{m-1}_{m-3,n-2} \cap K$ to which trees have been so attached, and so on. Since attaching trees in this way leaves the rank intact, we arrive at the theorem by induction on m. $\square$

Acknowledgement

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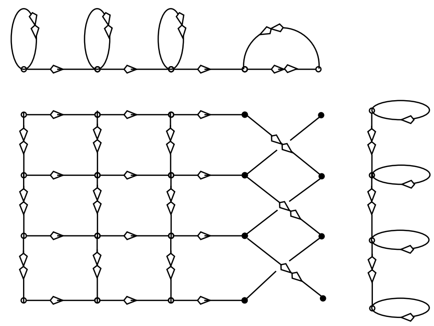

Figure 2: H, K, and $H \cap K$ when $k = m - 2$, $\ell = n - 2$

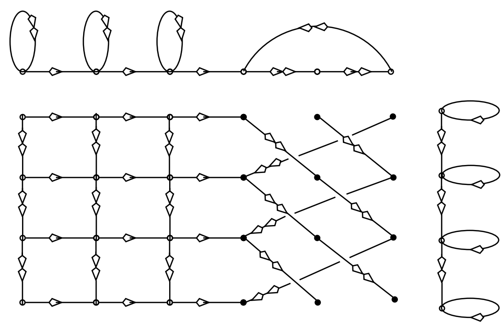

Figure 3: H, K, and $H \cap K$ when $k = m - 2$, $\ell = n - 3$

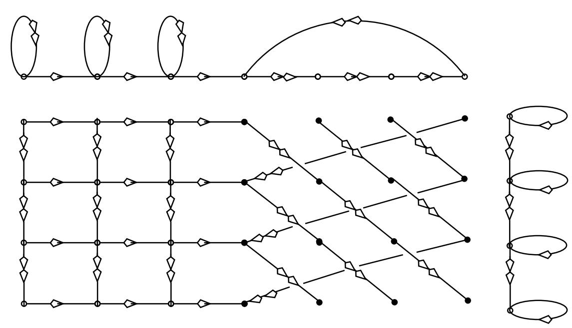

Figure 4: H, K, and $H \cap K$ when $k = m - 2$, $\ell = 0$

References

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