

A fake Schottky group in $\text{Mod}(S)$

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ABSTRACT. We use the classical construction of Schottky groups in hyperbolic geometry to produce non-Schottky subgroups of the mapping class group.

1. Introduction

In hyperbolic geometry, a Schottky group is a free convex cocompact Kleinian group, classically constructed as follows. Pick four pairwise disjoint closed balls $B_1^-, B_2^-, B_1^+, B_2^+$ in S_∞^{n-1} , the ideal boundary of hyperbolic n -space. Suppose there are isometries f_1 and f_2 so that

$$f_i(B_i^-) = \overline{S_\infty^{n-1} - B_i^+}.$$

Then $\langle f_1, f_2 \rangle$ is a Schottky group isomorphic to the free group F_2 of rank two.

Now let S be a closed surface of genus $g \geq 2$ and let $\text{Mod}(S) = \pi_0(\text{Homeo}^+(S))$ be its mapping class group. By way of analogy with the theory of Kleinian groups, B. Farb and L. Mosher defined [FM] a notion of convex cocompactness for subgroups of $\text{Mod}(S)$. In this setting, a *Schottky group* is a free convex cocompact subgroup of $\text{Mod}(S)$. In [KL1, KL2], we extended Farb and Mosher's analogy, providing several characterizations of convex cocompactness borrowed from the Kleinian setting (see also Hamenstädt [H]). The analogy is an imperfect one, see [KL3] and the references there, and we point out some new imperfections here.

THEOREM 1.1. *There exist pseudo-Anosov elements f_1 and f_2 in $\text{Mod}(S)$ and pairwise disjoint closed balls $B_1^-, B_2^-, B_1^+, B_2^+$ in $\mathbb{P}\mathcal{ML}(S)$ for which*

$$f_i(B_i^-) = \overline{\mathbb{P}\mathcal{ML}(S) - B_i^+}$$

and yet $\langle f_1, f_2 \rangle \cong F_2$ is not a Schottky group.

The construction is based on work of N. Ivanov, and it is clear from his work in [I] that he was aware of this construction (see also McCarthy [Mc]). The group $G = \langle f_1, f_2 \rangle$ contains reducible elements and so fails to be convex cocompact. It is worth noting that there are sufficiently high powers of the f_i that generate a Schottky group, as proven by Farb and Mosher [FM], see also [KL1, H].

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Part of the analogy between Kleinian groups and mapping class groups was developed by J. McCarthy and A. Papadopoulos [MP], who constructed a limit set Λ_G and domain of discontinuity $\Delta_G \subset \mathbb{P}\mathcal{ML}(S) - \Lambda_G$ for any subgroup $G < \text{Mod}(S)$, see Section 4. Unlike in the Kleinian setting, $\Delta_G \neq \mathbb{P}\mathcal{ML}(S) - \Lambda_G$ in general. While examples illustrate the necessity of taking an open set strictly smaller than $\mathbb{P}\mathcal{ML}(S) - \Lambda_G$ as a domain of discontinuity, it is not clear that Δ_G is an optimal choice. In [KL1], we asked whether or not Δ_G is the largest open set on which G acts properly discontinuously—see Question 3 there. Here, we answer this in the negative.

There is an obvious open set on which our group $G = \langle f_1, f_2 \rangle$ acts properly discontinuously and cocompactly, namely

$$\Omega = \bigcup_{g \in G} g \cdot \Theta$$

where Θ is the closure of the complement of our four balls. To see that Ω is open, note that Θ is contained in the interior \mathcal{U} of

$$f_1(\Theta) \cup f_1^{-1}(\Theta) \cup f_2(\Theta) \cup f_2^{-1}(\Theta)$$

and that

$$\Omega = \bigcup_{g \in G} g \cdot \mathcal{U}$$

If the f_i are chosen carefully, the set Ω will contain Δ_G properly, and we have the following theorem.

THEOREM 1.2. *There are irreducible subgroups $G < \text{Mod}(S)$ for which Δ_G is not the largest open set on which G acts properly discontinuously.*

Asymmetry of the construction provides another domain Ω' on which G acts properly discontinuously, and we will also show that G does not act properly discontinuously on the union $\Omega \cup \Omega'$.

Though Δ_G is not a maximal domain of discontinuity, we show in Section 5 that, for the groups in Theorem 1.2, it is nonetheless the intersection of all such maximal domains.

2. Surface dynamics

If \mathcal{X} is a subset of $\mathbb{P}\mathcal{ML}(S)$, we let

$$Z\mathcal{X} = \{[\nu] \in \mathbb{P}\mathcal{ML}(S) \mid i(\nu, \mu) = 0 \text{ for some } [\mu] \in \mathcal{X}\}$$

be the *zero locus* of \mathcal{X} . If $\mathcal{X} = \{[x]\}$ we sometimes write Zx for $Z\mathcal{X}$.

If f is pseudo-Anosov, then it acts with *north-south* dynamics on $\mathbb{P}\mathcal{ML}(S)$, meaning that it has unique attracting and repelling fixed points $[\mu_f^+]$ and $[\mu_f^-]$, respectively—all other points are attracted to $[\mu_f^+]$ under iteration of f . In fact, for any neighborhood U of $[\mu_f^+]$ and any compact set $K \subset \mathbb{P}\mathcal{ML}(S) - \{[\mu_f^-]\}$, there is a natural number N so that

$$(2.1) \quad f^n(K) \subset U$$

for any $n \geq N$.

Ivanov proves that there is a similar situation for most pure reducible elements (see the Appendix of [I]). In particular, suppose α is a nonseparating simple closed curve in S preserved by a mapping class ϕ that is pseudo-Anosov when restricted

to $S - \alpha$. Let $[\mu_\phi^+]$ and $[\mu_\phi^-]$ be the stable and unstable laminations for ϕ in $S - \alpha$ considered laminations on S , and note that

$$Z\mu_\phi^- = \{[s\mu_\phi^- + (1-s)\alpha] \in \mathbb{P}\mathcal{ML}(S) \mid s \in [0, 1]\}.$$

If $K \subset \mathbb{P}\mathcal{ML}(S) - Z\mu_\phi^-$ is a compact set and U a neighborhood of $[\mu_\phi^+]$, then there is an $N > 0$ such that for all $n \geq N$ we have

$$(2.2) \quad \phi^n(K) \subset U.$$

Given a mapping class g of either type above, let $\lambda(g)$ denote the *expansion factor* of g , the number such that

$$g(\mu_g^+) = \lambda(g)\mu_g^+.$$

3. The construction

Let α be a nonseparating curve fixed by a mapping class ϕ that is pseudo-Anosov on $S - \alpha$, and let $[\mu_\phi^+]$ and $[\mu_\phi^-]$ be as in the previous section.

Let $\mathbb{S}_\phi \subset \mathbb{P}\mathcal{ML}(S)$ be a bicollared $(6g-8)$ -dimensional sphere dividing $\mathbb{P}\mathcal{ML}(S)$ into two closed balls \mathcal{A}_ϕ and \mathcal{B}_ϕ containing $Z\mu_\phi^-$ and $[\mu_\phi^+]$, respectively.

According to (2.2), there is an $N > 0$ so that for all $n \geq N$ we have

$$\phi^n(\mathcal{B}_\phi) \subset \text{int}(\mathcal{B}_\phi).$$

So we choose an $n \geq N$, let $h = \phi^n$, $B_h^- = \mathcal{A}_\phi$, and $B_h^+ = h(\mathcal{B}_\phi)$.

Recall H. Masur's theorem [Ma] that the set

$$\{([\mu_\psi^+], [\mu_\psi^-]) \mid \psi \in \text{Mod}(S) \text{ pseudo-Anosov}\}$$

is dense in $\mathbb{P}\mathcal{ML}(S) \times \mathbb{P}\mathcal{ML}(S)$. So we choose a pseudo-Anosov ψ whose fixed points $[\mu_\psi^+]$ and $[\mu_\psi^-]$ lie in $\mathbb{P}\mathcal{ML}(S) - (B_h^- \cup B_h^+)$. We let $\mathbb{S}_\psi \subset \mathbb{P}\mathcal{ML}(S) - (B_h^- \cup B_h^+)$ be a bicollared $(6g-8)$ -sphere which bounds two balls: $\mathcal{A}_\psi \subset \mathbb{P}\mathcal{ML}(S) - (B_h^- \cup B_h^+)$ containing $[\mu_\psi^-]$ and \mathcal{B}_ψ containing $[\mu_\psi^+]$. As $\mathbb{P}\mathcal{ML}(S) - (B_h^- \cup B_h^+)$ is a neighborhood of $[\mu_\psi^+]$, (2.1) provides an $M > 0$ such that for all $m \geq M$, we have $\psi^m(\mathcal{B}_\psi) \subset \mathbb{P}\mathcal{ML}(S) - (B_h^- \cup B_h^+)$. Arguing as in [I], we may choose m so that $\psi^m h$ is pseudo-Anosov, and we do so. We let $f = \psi^m$, $B_f^- = \mathcal{A}_\psi$, and $B_f^+ = f(\mathcal{B}_\psi)$.

We now have elements f , h , and pairwise disjoint closed balls

$$B_h^-, B_h^+, B_f^-, B_f^+$$

with

$$h(B_h^-) = \overline{\mathbb{P}\mathcal{ML}(S) - B_h^+} \text{ and } f(B_f^-) = \overline{\mathbb{P}\mathcal{ML}(S) - B_f^+}.$$

See Figure 1.

Let $G = \langle f, h \rangle$, set

$$\Theta = \overline{\mathbb{P}\mathcal{ML}(S) - (B_h^- \cup B_h^+ \cup B_f^- \cup B_f^+)}.$$

and let

$$\Omega = \bigcup_{g \in G} g \cdot \Theta.$$

The group G acts on Ω properly discontinuously and cocompactly with fundamental domain Θ , and the usual ping-pong argument implies that $G \cong F_2$.

A slight modification now provides the desired example.

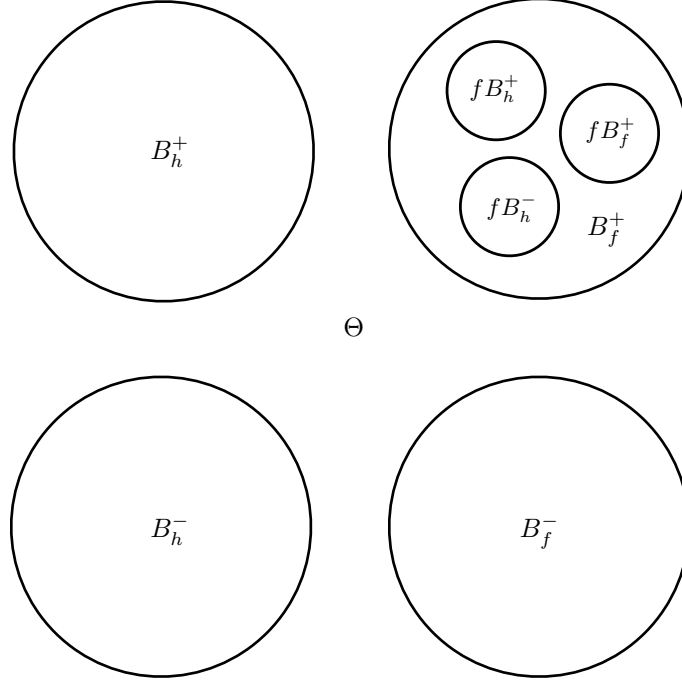


FIGURE 1

We let $f_1 = fh$ and $f_2 = f$, both pseudo-Anosov by construction. Of course, $G = \langle f_1, f_2 \rangle$, and we need only find balls B_1^\pm and B_2^\pm with

$$(3.1) \quad f_i(B_i^-) = \overline{\mathbb{PML}(S) - B_i^+}.$$

Set $B_1^- = B_h^-$ and $B_1^+ = f(B_h^+)$. The ball B_2^- is constructed as a regular neighborhood of $B_f^- \cup B_h^+ \cup \delta$ in $\overline{\mathbb{PML}(S) - (B_h^- \cup B_f^+)}$, where δ is an arc in Θ from B_f^- to B_h^+ . The ball B_2^+ is defined to be $\overline{\mathbb{PML}(S) - f(B_2^-)}$. See Figure 2.

One can now check (3.1).

4. Proper discontinuity

Let $G = \langle h, f \rangle$ be the group constructed in the previous section, and let ∂G be the Gromov boundary of G . By the work in [MP], the limit set

$$\Lambda_G = \overline{\{[\mu_g^+] \in \mathbb{PML}(S) \mid g \in G \text{ is pseudo-Anosov}\}}$$

is the unique minimal closed G -invariant subset of $\mathbb{PML}(S)$. In [KL2] we showed that one may choose h and f as above so that G has the following property.

PROPERTY 4.1. *There exists a continuous G -equivariant homeomorphism*

$$\mathfrak{J}: \partial G \rightarrow \Lambda_G.$$

Moreover, for each $x \in \partial G$ which is a fixed point of a conjugate ${}^g h$ of h , $\mathfrak{J}(x)$ is the stable or unstable lamination of that conjugate ${}^g h$ (respecting the dynamics).

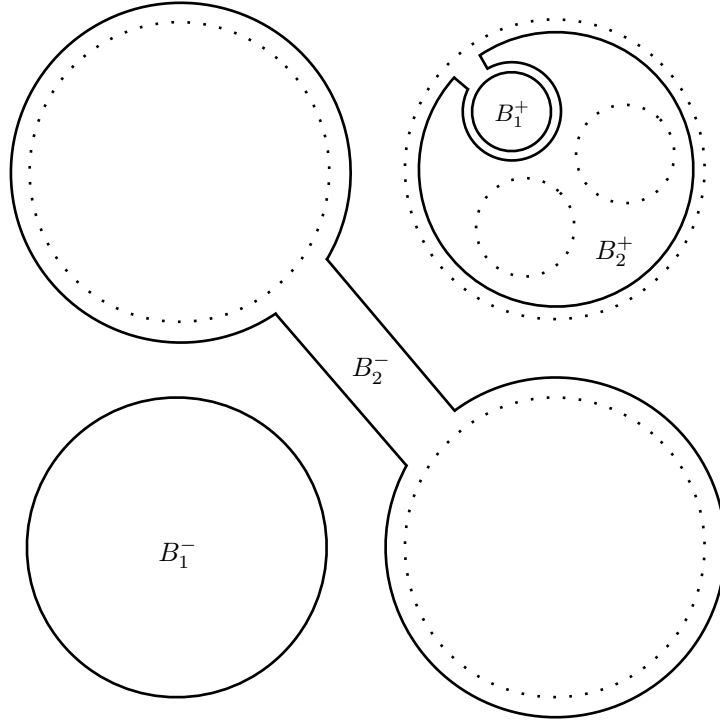


FIGURE 2

Otherwise $\mathfrak{J}(x)$ is a uniquely ergodic filling lamination. In particular, every element $g \in G$ not conjugate to a power of h is pseudo-Anosov.

We henceforth assume that G satisfies Property 4.1.
 The domain of discontinuity is defined to be

$$\Delta_G = \mathbb{P}\mathcal{ML}(S) - Z\Lambda_G.$$

This is an open set on which G acts properly discontinuously [MP], which justifies the name.

We may describe the zero locus $Z\Lambda_G$ for G explicitly. For each conjugate ${}^g h$ of h , we have the attracting and repelling fixed points $x_{g^{\pm}h}$ in ∂G . By Property 4.1, the map \mathfrak{J} sends these to the stable and unstable laminations

$$\mathfrak{J}(x_{g^{\pm}h}^{\pm}) = [\mu_{g^{\pm}h}^{\pm}] = g[\mu_h^{\pm}].$$

For any such point $g[\mu_h^{\pm}] \in \Lambda_G$, the set $Zg\mu_h^{\pm} = gZ\mu_h^{\pm}$ is a 1-simplex in $Z\Lambda_G$. Since $\mathfrak{J}(x)$ is uniquely ergodic and filling for every other point $x \in \partial G$, it follows that $Z\Lambda_G$ is the union of Λ_G and all of these intervals.

The intervals $Z\mu_h^-$ and $Z\mu_h^+$ intersect each other at α , and so the union

$$\mathbb{J}_h = Z\mu_h^- \cup Z\mu_h^+$$

is an interval joining μ_h^- to μ_h^+ . All in all, we have

$$(4.1) \quad Z\Lambda_G = \Lambda_G \cup \bigcup_{g \in G} g\mathbb{J}_h$$

We impose one further restriction on h and f —more precisely, on the balls B_f^\pm . Since the fixed points of f do not meet the interval \mathbb{J}_h , we may replace f with a power so that the balls B_f^\pm are disjoint from this interval. This implies that

$$\text{int } Z\mu_h^+ = Z\mu_h^+ \cap \bigcup_{n \in \mathbb{Z}} h^n \Theta$$

and so $Z\mu_h^+$ intersects the h^n translates of Θ , and no other G -translates. As $Z\mu_h^-$ does not intersect Ω , these are the only G -translates of Θ that \mathbb{J}_h intersects. Write $\Sigma_h^\pm = \partial B_h^\pm$ and $\Sigma_f^\pm = \partial B_f^\pm$.

We claim that

$$\Sigma_f^+ \cap Z\Lambda_G = \emptyset.$$

To see this, note that if Σ_f^+ nontrivially intersected $Z\Lambda_G$, it would do so in some $g\mathbb{J}_h$, by (4.1); and then g must be a power of h , since Σ_f^+ lies in Θ . But $h\mathbb{J}_h = \mathbb{J}_h$, and so Σ_f^+ would intersect \mathbb{J}_h , contrary to our choice of f . The claim follows.

Now, Theorem 1.2 will follow from

THEOREM 4.2. *The set Δ_G is properly contained in Ω . In fact,*

$$\Omega = \mathbb{P}\mathcal{M}\mathcal{L}(S) - \left(\Lambda_G \cup \bigcup_{g \in G} gZ\mu_h^- \right).$$

First note that $\Delta_G \neq \Omega$ as $\Sigma_h^- \subset \Theta \subset \Omega$ nontrivially intersects $\mathbb{J}_h \subset Z\Lambda_G = \mathbb{P}\mathcal{M}\mathcal{L}(S) - \Delta_G$.

To prove the containment, we must gather some information about the complement of Ω . Let $\mathfrak{X} = \mathbb{P}\mathcal{M}\mathcal{L}(S) - \Omega$.

LEMMA 4.3. *There is a continuous G -equivariant map*

$$\mathfrak{K}: \mathfrak{X} \rightarrow \partial G.$$

PROOF. The spheres Σ_h^\pm and Σ_f^\pm are bicollared with collars $N(\Sigma_h^\pm)$ and $N(\Sigma_f^\pm)$. We assume as we may that

$$h(N(\Sigma_h^-)) = N(\Sigma_h^+) \text{ and } f(N(\Sigma_f^-)) = N(\Sigma_f^+)$$

and that all of the G -translates of these collars are pairwise disjoint.

Let \mathcal{G} be the Cayley graph of G and identify $\partial G = \partial \mathcal{G}$. We define a continuous G -equivariant map

$$\mathfrak{K}_0: \Omega \rightarrow \mathcal{G}$$

by identifying \mathcal{G} with the tree dual to the hypersurface

$$\bigcup_{g \in G} g(\Sigma_h^-) \cup \bigcup_{g \in G} g(\Sigma_f^-)$$

in Ω and projecting in the usual manner, see [Sh].

The map \mathfrak{K}_0 extends continuously to a G -equivariant map

$$\mathfrak{K}: \mathbb{P}\mathcal{M}\mathcal{L}(S) \rightarrow \bar{\mathcal{G}} = \mathcal{G} \cup \partial G$$

whose restriction to \mathfrak{X} is the map we desire. The extension is described concretely as follows.

First note that given any point $[\eta] \in \mathfrak{X}$, there is a unique sequence of elements

$$x_1^{\epsilon_1}, x_2^{\epsilon_2}, x_3^{\epsilon_3}, \dots$$

where $x_i \in \{f, h\}$ and $\epsilon_i \in \{\pm 1\}$ with the property that $[\eta]$ is contained in the nested intersection

$$\bigcap_{i=1}^{\infty} x_1^{\epsilon_1} \dots x_i^{\epsilon_i} (B_{x_i}^{\epsilon_i}).$$

where $B_h^{\pm 1} = B_h^{\pm}$ and $B_f^{\pm 1} = B_f^{\pm}$. Identifying ∂G with the set of infinite reduced words, our map is given there by

$$\mathfrak{K}([\eta]) = x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3} \dots$$

To see that \mathfrak{K} is continuous, let $\mathcal{U}_g \subset \bar{\mathcal{G}}$ be the open set consisting of all infinite reduced words in ∂G with prefix g together with the union of the open tails of the corresponding paths in \mathcal{G} . Now if g ends in $x_0^{\epsilon_0}$ with $x_0 \in \{h, f\}$ and $\epsilon_0 \in \{\pm 1\}$, then

$$\mathfrak{K}^{-1}(\mathcal{U}_g) = g x_0^{-\epsilon_0} (\text{int } B_{x_0}^{\epsilon_0})$$

which is open. \square

LEMMA 4.4. \mathfrak{K} is a one-sided inverse to \mathfrak{J} . That is, $\mathfrak{K} \circ \mathfrak{J} = \text{id}_{\partial G}$.

PROOF. Since \mathfrak{X} is a G -invariant closed set, it contains Λ_G , and so $\mathfrak{K} \circ \mathfrak{J}$ is well-defined. Next, suppose that x_f^+ is the attracting fixed point of f . Then $\mathfrak{J}(x_f^+) = [\mu_f^+]$ is the attracting fixed point in $\mathbb{P}\mathcal{ML}(S)$ of f , and hence $\mathfrak{K}(\mathfrak{J}(x_f^+)) = x_f^+$. The same is true for any conjugate of f , and hence $\mathfrak{K} \circ \mathfrak{J}$ is the identity on the set of attracting fixed points of conjugates of f . Being G -invariant, this set is dense in ∂G , and so, by continuity, $\mathfrak{K} \circ \mathfrak{J}$ is the identity on all of ∂G . \square

Theorem 4.2 follows easily from the following lemma.

LEMMA 4.5. For all $x \in \partial G$, we have $\mathfrak{K}^{-1}(x) \subset Z\mathfrak{J}(x)$. In fact, if x is the repelling fixed point $x_{g_h^-}$ of a conjugate g_h of h , then $\mathfrak{K}^{-1}(x) = gZ\mu_h^-$. Otherwise, the set $\mathfrak{K}^{-1}(x)$ is a singleton contained in Λ_G .

PROOF OF THEOREM 4.2 ASSUMING LEMMA 4.5. By the first statement, $\mathfrak{X} \subset Z\Lambda_G$ since

$$Z\Lambda_G = \bigcup_{x \in \partial G} Z\mathfrak{J}(x).$$

So $\Omega \supset \Delta_G$ as required. Again, the containment is proper as $Z\mu_h^+$ nontrivially intersects Ω .

The description of Ω follows from the second and third statements. \square

We need the following general fact about sequences of laminations.

LEMMA 4.6. Suppose $\mathfrak{S} \subset \mathcal{ML}(S)$ is a compact set, $\{f_k\} \subset \text{Mod}(S)$ is an infinite sequence of distinct pseudo-Anosov mapping classes with

$$\mu_{f_k}^{\pm} \rightarrow \mu^{\pm}$$

in $\mathcal{ML}(S)$, and that $\{\nu_k\}_{k=1}^{\infty} \subset \mathfrak{S}$ and $\{t_k\}_{k=1}^{\infty} \subset \mathbb{R}_+$ are sequences with

$$t_k f_k(\nu_k) \rightarrow \eta$$

in $\mathcal{ML}(S)$.

If there is an $r > 0$ such that

$$i(\nu, \mu^\pm) > r$$

for all $\nu \in \mathfrak{S}$, then $t_k \rightarrow 0$.

PROOF. Note that continuity of i and compactness of \mathfrak{S} imply that there exist $K > 0$ and $R > 1$ such that for all $k \geq K$ and all $\nu \in \mathfrak{S}$

$$\frac{1}{R} < i(\nu, \mu_{f_k}^\pm) < R.$$

By the continuity of i we have

$$\lim_{k \rightarrow \infty} i(t_k f_k(\nu_k), \mu_{f_k}^-) = i(\eta, \mu^-),$$

and so, for sufficiently large k , we have

$$i(\eta, \mu^-) - 1 < i(t_k f_k(\nu_k), \mu_{f_k}^-) < i(\eta, \mu^-) + 1.$$

The central term of this inequality is also given by

$$\begin{aligned} i(t_k f_k(\nu_k), \mu_{f_k}^-) &= i(t_k \nu_k, f_k^{-1}(\mu_{f_k}^-)) \\ &= t_k i(\nu_k, \lambda(f_k^{-1})\mu_{f_k}^-) \\ &= t_k \lambda(f_k) i(\nu_k, \mu_{f_k}^-) \end{aligned}$$

where $\lambda(f_k)$ is the expansion factor of f_k , and so, for all sufficiently large k , we have

$$\frac{i(\eta, \mu^-) - 1}{R} < t_k \lambda(f_k) < R(i(\eta, \mu^-) + 1).$$

Since the f_k are all distinct, and their fixed points converge in $\mathbb{P}\mathcal{ML}(S)$, it follows that $\lambda(f_k) \rightarrow \infty$. So $t_k \rightarrow 0$ as required. \square

PROOF OF LEMMA 4.5. First assume that $x \in \partial G$ is the fixed point of a conjugate of h . By the G -equivariance of \mathfrak{R} , it suffices to consider the case of h itself. Then, we have $x = x_h^+$ or $x = x_h^-$. In this case, the sequences of balls nesting to $\mathfrak{R}^{-1}(x_h^+)$ and $\mathfrak{R}^{-1}(x_h^-)$ are given by

$$\{h^k(B_h^+)\}_{k=1}^\infty \text{ and } \{h^{-k}(B_h^-)\}_{k=1}^\infty,$$

respectively.

From the discussion in Section 2, we already know that

$$\mathfrak{R}^{-1}(x_h^+) = \bigcap_{k=1}^\infty h^k(B_h^+) = \{[\mu_h^+]\} \subset Z\mathfrak{I}(x_h^+)$$

and

$$\mathfrak{R}^{-1}(x_h^-) = \bigcap_{k=1}^\infty h^{-k}(B_h^-) = Z\mu_h^- = Z\mathfrak{I}(x_h^-).$$

If $g \in G$ is any other element not conjugate to a power of h , then, by Property 4.1, g is pseudo-Anosov, and the dynamical properties of pseudo-Anosov mapping classes discussed in Section 2 implies

$$\mathfrak{R}^{-1}(x^\pm(g)) = \{[\mu^\pm(g)]\} = Z\mathfrak{I}(x^\pm(g)).$$

Therefore, to complete the proof of the lemma, we assume that $x \in \partial G$ is not a fixed point of any element of G .

We write x as an infinite reduced word

$$x = x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3} \cdots .$$

Since x is not the fixed point of any element of G , we can assume that $x_n = f$ and, say, $\epsilon_n = +1$ for infinitely many n (the case that $x_n = f$ and $\epsilon_n = -1$ for infinitely many n is similar). The G -equivariance of \mathfrak{R} implies that we may also assume that $x_1 = f$ and $\epsilon_1 = 1$. Let $\{n_k\}_{k=1}^{\infty}$ be the increasing sequence of natural numbers for which $x_{n_k} = f$ and $\epsilon_{n_k} = +1$. Finally, set

$$f_k = x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3} \cdots x_{n_k}^{\epsilon_{n_k}} \in G.$$

Then, we have $\mathfrak{R}^{-1}(x)$ expressed as the nested intersection

$$\mathfrak{R}^{-1}(x) = \bigcap_{k=1}^{\infty} f_k(B_f^+).$$

Any point $[\eta]$ in the frontier of $\mathfrak{R}^{-1}(x)$ is a limit of a sequence in the frontiers

$$[\eta] = \lim_{k \rightarrow \infty} f_k([\nu_k])$$

where

$$[\nu_k] \in \text{Fr}(B_f^+) = \Sigma_f^+.$$

We fix any such $[\eta] \in \text{Fr}(\mathfrak{R}^{-1}(x))$ and such a sequence $\{[\nu_k]\}$.

We pass to a further subsequence so that $\mu_{f_k}^{\pm} \rightarrow \mu^{\pm} \in \mathcal{ML}(S)$. Since $[\mu_{f_k}^{\pm}] \in \Lambda_G$ for all k , we also have $[\mu^{\pm}] \in \Lambda_G$. In fact, since $f_k = x_1^{\epsilon_1} \cdots x_{n_k}^{\epsilon_{n_k}}$ is cyclically reduced, the axes for f_k in \mathcal{G} all go through the origin and limit to a geodesic $\gamma \subset \mathcal{G}$ through $\mathbf{1}$ with positive ray ending at x . Therefore, $x_{f_k}^+ \rightarrow x$ as $k \rightarrow \infty$, and by continuity of \mathfrak{J} , it follows that

$$\mathfrak{J}(x) = [\mu^+] \in \Lambda_G.$$

Moreover, the negative ray of γ ends at some point $y \in \partial G$ and is described by an infinite word

$$y = y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3} \cdots$$

where $y_1^{\delta_1} \neq f$ since $x_1^{\epsilon_1} = f$ and γ is a geodesic. Therefore, again appealing to the continuity of \mathfrak{J} we see that

$$\mathfrak{J}(y) = [\mu^-] \in \Lambda_G \cap \mathbb{P}\mathcal{ML}(S) - B_f^+.$$

By similar reasoning, for any $[\mu] \in \Lambda_G \cap B_f^+$, we have

$$f_k([\mu]) \rightarrow [\mu^+] = \mathfrak{J}(x).$$

In fact, it follows from [MP, Lemma 2.7] that there is a μ (a fixed point of a pseudo-Anosov in G) and a sequence s_k tending to zero such that

$$\lim_{k \rightarrow \infty} s_k f_k(\mu) = \mu^+ \in \mathcal{ML}(S).$$

We now let $\mathfrak{S} \subset \mathcal{ML}(S)$ be the image of Σ_f^+ under some continuous section of $\mathcal{ML}(S) \rightarrow \mathbb{P}\mathcal{ML}(S)$. Since $\Sigma_f^+ \cap Z\Lambda_G = \emptyset$, there is an $r > 0$ such that

$$i(\nu, \mu^{\pm}) > r$$

for every $\nu \in \mathfrak{S}$.

We take the representatives ν_k of $[\nu_k]$ to lie in \mathfrak{S} . Then, according to Lemma 4.6, the sequence t_k for which

$$\lim_{k \rightarrow \infty} t_k f_k(\nu_k) = \eta$$

must tend to zero. So

$$i(\eta, \mu^+) = \lim_{k \rightarrow \infty} i(t_k f_k(\nu_k), s_k f_k(\mu)) = \lim_{k \rightarrow \infty} t_k s_k i(\nu_k, \mu) = 0$$

since s_k and t_k tend to zero and $i(\nu_k, \mu)$ is uniformly bounded by compactness of \mathfrak{S} . Since μ^+ is uniquely ergodic, we conclude that $[\eta] = [\mu^+] = \mathfrak{J}(x)$.

This means that the frontier of $\mathfrak{R}^{-1}(x)$ is precisely $\{\mathfrak{J}(x)\}$, and hence

$$\mathfrak{R}^{-1}(x) = \{\mathfrak{J}(x)\} = Z\mathfrak{J}(x)$$

as required. \square

5. Final comments

If we replace h with h^{-1} in our construction we obtain another G -invariant open set Ω' on which G acts properly discontinuously and cocompactly. By Lemma 4.5, we have descriptions

$$\Omega = \mathbb{P}\mathcal{M}\mathcal{L}(S) - \left(\Lambda_G \cup \bigcup_{g \in G} gZ\mu_h^- \right)$$

and

$$\Omega' = \mathbb{P}\mathcal{M}\mathcal{L}(S) - \left(\Lambda_G \cup \bigcup_{g \in G} gZ\mu_h^+ \right),$$

and it follows that

$$\Omega \cup \Omega' = \mathbb{P}\mathcal{M}\mathcal{L}(S) - \left(\Lambda_G \cup \bigcup_{g \in G} G \cdot \alpha \right).$$

The group G does not act properly discontinuously on $\Omega \cup \Omega'$, and in fact, we have the following.

PROPOSITION 5.1. *If $\mathcal{U} \subset \mathbb{P}\mathcal{M}\mathcal{L}(S)$ is any open set on which G acts properly discontinuously, then $\mathcal{U} \subset \Omega$ or $\mathcal{U} \subset \Omega'$.*

PROOF. Let $\mathcal{U} \subset \mathbb{P}\mathcal{M}\mathcal{L}(S)$ be a G -invariant open set. We will show that if \mathcal{U} is not contained in either Ω or Ω' , then G does not act properly discontinuously on \mathcal{U} .

If $\mathcal{U} \cap \Lambda_G \neq \emptyset$, then since G acts minimally on Λ_G and \mathcal{U} is G -invariant, we must have $\Lambda_G \subset \mathcal{U}$. As G clearly fails to act properly discontinuously on \mathcal{U} in this case, we assume that $\mathcal{U} \cap \Lambda_G = \emptyset$.

So if \mathcal{U} fails to be contained in either Ω or Ω' , there are points $[\eta^+] \in \mathcal{U} \cap Z\mu_h^+$ and $[\eta^-] \in \mathcal{U} \cap Z\mu_h^-$. Moreover, $[\eta^\pm]$ is in the interior of $Z\mu_h^\pm$. Let Υ^\pm be small compact balls contained in \mathcal{U} containing $[\eta^\pm]$. Since $[\eta^+] \in \Omega$, we may assume that $\Upsilon^+ \subset \Omega$. Moreover, G -invariance of \mathcal{U} allows us to pick $[\eta^+]$ and Υ^+ to lie in B_h^- .

After passing to a subsequence, we can assume that the sequence of sets $\{h^{-k_j}(\Upsilon^+)\}_{j=1}^\infty$ converges in the Hausdorff topology. Moreover, we have

$$\lim_{j \rightarrow \infty} h^{-k_j}(\Upsilon^+) \subset \bigcap_{k=1}^{\infty} h^{-k}(B_k^-) = Z\mu_h^-.$$

Note that the Hausdorff limit must be connected since Υ^+ is. This limit contains α as the pointwise limit of $h^{-k}[\eta^+]$, and $[\mu_h^-]$ as the pointwise limit of any other point of Υ^+ under h^{-k} . Therefore,

$$\lim_{j \rightarrow \infty} h^{-k_j}(\Upsilon^+) = Z\mu_h^-.$$

Now, consider the compact set $\Upsilon = \Upsilon^+ \cup \Upsilon^-$. Since $\text{int}(\Upsilon^-)$ is a neighborhood of $[\eta^-]$, we have

$$h^{-k_j}(\Upsilon) \cap \Upsilon \supset h^{-k_j}(\Upsilon^+) \cap \text{int}(\Upsilon^-) \neq \emptyset$$

for all sufficiently large j . So G does not act properly discontinuously on \mathcal{U} . \square

From this we deduce that Ω and Ω' are the only maximal open sets on which G acts properly discontinuously. By our descriptions of Ω and Ω' we also have

$$\Delta_G = \Omega \cap \Omega'.$$

It follows that Δ_G can be described purely in terms of the action of G on $\mathbb{P}\mathcal{ML}(S)$, *without referring to geometric structures on the surface*.

Though Δ_G may not be a *maximal* open set on which G acts nicely, it remains a *canonically defined* one, and we pose the following question.

QUESTION 5.2. If G is an irreducible subgroup of $\text{Mod}(S)$, is Δ_G the intersection of all maximal open sets on which G acts properly discontinuously?

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