Dimer Problems.

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Abstract

The dimer model is a statistical mechanical model on a graph, where configurations consist of perfect matchings of the vertices. For planar graphs, expressions for the partition function and local statistics can be obtained using determinants. The planar dimer model can be used to model a number of other statistical mechanical processes such as the planar Ising model and free fermions. It is also a model for crystal surfaces in $\mathbb{R}^3$.
keywords: exactly solved model, free fermions, domino tiling, perfect matching, Pfaffian, height function, Gaussian free field, conformal invariance, spanning trees.

1 Definitions

The dimer model arose in the mid-20th century as an example of an exactly solvable statistical mechanical model in two dimensions with a phase transition. It is used to model a number of physical processes: free fermions in 1 dimension, the 2-dimensional Ising model, and various other two-dimensional statistical-mechanical models at restricted parameter values, such as the 6- and 8-vertex models and $O(n)$-models. A number of observable quantities such as the "height function" and densities of motifs have been shown to have conformal invariance properties in the scaling limit (when the lattice spacing tends to zero).

Recently the model is also used as an elementary model of crystalline surfaces in $\mathbb{R}^3$.

A *dimer covering*, or *perfect matching*, of a graph is a set of edges ("dimers") which covers every vertex exactly once. In other words it is a pairing of adjacent vertices; see Figure 1a which is a dimer covering of an 8×8 grid. Dimer coverings of a grid are sometimes represented as domino tilings, that is, tilings with 2×1 rectangles (Figure 1b). The *dimer model* is the study of the set of dimer coverings of a

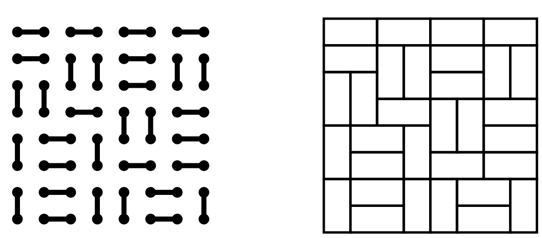

Figure 1: A dimer covering of a grid and the corresponding domino tiling.

graph. Historically the underlying graph is taken to be a regular lattice in two dimensions, for example the square grid or the honeycomb lattice, or a finite part of such a lattice.

Dimer coverings of the honeycomb graph are in bijection with tilings of plane regions with 60° rhombi, also known as lozenges, see Figure 2. These tilings in turn are projections of piecewise-linear surfaces in

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$\mathbb{R}^3$ composed of unit squares in the 2-skeleton of $\mathbb{Z}^3$. So one can think of honeycomb dimer coverings as modelling discrete surfaces in $\mathbb{R}^3$. These surfaces are *monotone* in the sense that the orthogonal projection to the plane $P_{111} = \{(x, y, z) \mid x + y + z = 0\}$ is injective.

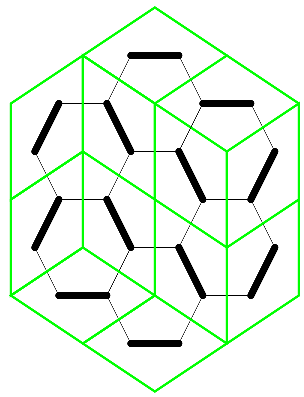

Figure 2: Honeycomb dimers (solid) and the corresponding "lozenge" tilings (gray).

Other models related to the dimer model are:

- The spanning tree model on planar graphs. The set of spanning trees on a planar graph is in bijection with the set of dimer coverings on an associated bipartite planar graph. Conversely, dimer coverings of a bipartite planar graph are in bijection with directed spanning trees on an associated graph.
- The Ising model on a planar graph with zero external field can be modeled with dimers on an associated planar graph.
- Plane partitions (three-dimensional versions of integer partitions). Viewing a plane partition along the $(1, 1, 1)$-direction, one sees a lozenge tiling of the plane.
- Annihilating random walks in one dimension can be modelled with dimers on an associated planar graph.
- the monomer-dimer model, where one allows a certain density of holes (monomers) in a dimer covering. This model is unsolved at present, although some partial results have been obtained.

1.1 Gibbs measures

The most general setting in which the dimer model can be solved is that of an arbitrary planar graph with energies on the edges. We define here the corresponding measure.

Let $G = (V, E)$ be a graph and $\mathcal{M}(G)$ the set of dimer coverings of G. Let $\mathcal{E}$ be a real-valued function on the edges of G, with $\mathcal{E}(e)$ representing the energy associated to a dimer on the bond e. One defines the energy of a dimer covering as the sum of the energies of those bonds covered with dimers. The *partition function* of the model on $(G, \mathcal{E})$ is then the sum

$$Z = \sum_{C \in \mathcal{M}(G)} e^{-\mathcal{E}(C)/kT}$$

where the sum is over dimer coverings. In what follows we will usually take $kT = 1$ for simplicity. Note that Z depends on both G and $\mathcal{E}$.

The partition function is well-defined for a finite graph and defines the *Gibbs measure*, which is by definition the probability measure $\mu = \mu_{\mathcal{E}}$ on the set $\mathcal{M}(G)$ of dimer coverings satisfying $\mu(C) = \frac{1}{Z} e^{-\mathcal{E}(C)}$ for a covering C.

For an infinite graph G with fixed energy function $\mathcal{E}$, a Gibbs measure on $\mathcal{M}(G)$ is by definition any measure which a limit of the Gibbs measures on a sequence of finite subgraphs which fill out G.

There may be many Gibbs measures on an infinite graph, since this limit typically depends on the sequence of finite graphs. When G is an infinite periodic graph (and $\mathcal{E}$ is periodic as well), it is natural to consider translation-invariant Gibbs measures; one can show that in the case of a bipartite, periodic planar graph the translation-invariant and ergodic Gibbs measures form a two-parameter family—see Theorem 3 below.

For a translation-invariant Gibbs measure ν which is a limit of Gibbs measures on an increasing sequence of finite graphs G_n, one can define the *partition function per vertex* of ν to be the limit

$$Z = \lim_{n\to\infty} Z(G_n)^{1/|G_n|},$$

where $|G_n|$ is the number of vertices of G_n. The *free energy*, or *surface tension*, of ν is $-\log Z$.

2 Combinatorics

2.1 Partition function

One can compute the partition function for dimer coverings on a finite planar graph G as the Pfaffian (square root of the determinant) of a certain anti-symmetric matrix, the *Kasteleyn matrix.* The Kasteleyn matrix is an oriented adjacency matrix of G, indexed by the vertices V: orient the edges of a graph embedded in the plane so that each face has an odd number of clockwise oriented edges. Then define $K = (K_{vv'})$ with

$$K_{vv'} = \pm e^{-\mathcal{E}(vv')}$$

if G has an edge vv', with a sign according to the orientation of that edge, and $K_{vv'} = 0$ if v, v' are not adjacent. We then have the following result of Kasteleyn:

Theorem 1. $Z = |\mathrm{Pf}(K)| = \sqrt{|\det K|}$.

Here $\mathrm{Pf}(K)$ denotes the Pfaffian of K.

Such an orientation of edges (which always exists for planar graphs) is called a Kasteleyn orientation; any two such orientations can be obtained from one another by a sequence of operations consisting of reversing the orientations of all edges at a vertex.

If G is a bipartite graph, that is, the vertices can be colored black and white with no neighbors having the same color, then the Pfaffian of K is the determinant of the submatrix whose rows index the white vertices and columns index the black vertices. For bipartite graphs, instead of orienting the edges one can alternatively multiply the edge weights by a complex number of modulus 1, with the condition that the alternating product around each face (the first, divided by the second, times the third, as so on) is real and negative.

For nonplanar graphs, one can compute the partition function as a sum of Pfaffians; for a graph embedded on a surface of Euler characteristic χ, this requires in general $2^{2-\chi}$ Pfaffians.

2.2 Local statistics

The inverse of the Kasteleyn matrix can be used to compute the *local statistics*, that is, the probability that a given set of edges occurs in a random dimer covering (random with respect to the Gibbs measure μ).

Theorem 2. *Let $S = \{(v_1, v_2), \dots, (v_{2k-1}, v_{2k})\}$ be a set of edges of G. The probability that all these edges occur in a μ-random covering is*

$$\Pr(S) = \left(\prod_{i=1}^{k} K_{v_{2i-1},v_{2i}}\right) \mathrm{Pf}_{2k\times 2k}((K^{-1})_{v_i,v_j}).$$

Again, for bipartite graphs the Pfaffian can be made into a determinant.

2.3 Heights

2.3.1 Bipartite graphs

Suppose G is a bipartite planar graph. A 1-*form* on G is simply a function on the set of oriented edges which is antisymmetric with respect to reversing the edge orientation: $f(-e) = -f(e)$ for an edge e. A 1-form can be identified with a flow: just flow by $f(e)$ along oriented edge e. The divergence of the flow f is then d^*f. Let Ω be the space of flows on edges of G, with divergence 1 at each white vertex and divergence -1 at each black vertex, and such that the flow along each edge from white to black is in $[0, 1]$. From a dimer covering M one can construct such a flow $\omega(M) \in \Omega$: just flow one unit along each dimer, and zero on the remaining edges. The set Ω is a convex polyhedron in $\mathbb{R}^E$ and its vertices can be seen to be exactly the dimer coverings.

Given any two flows $\omega_1, \omega_2 \in \Omega$, their difference is a divergence-free flow. Its dual $(\omega_1 - \omega_2)^*$ (or conjugate flow) defined on the planar dual of G is therefore the gradient of a function h on the faces of G, that is, $(\omega_1 - \omega_2)^* = dh$, where h is well defined up to an additive constant.

When ω_1 and ω_2 come from dimer coverings, h is integer valued, and is called the *height difference* of the coverings. The level sets of the function h are just the cycles formed by the union of the two matchings. If we fix a "base point" covering ω_0 and a face f_0 of G, we can then define the *height function* of any dimer covering (with flow ω) to be the function h with value zero at f_0 and which satisfies $dh = (\omega - \omega_0)^*$.

2.3.2 Non-bipartite graphs

On a nonbipartite planar graph the height function can be similarly defined modulo 2. Fix a base covering ω_0; for any other covering ω, the superposition of ω_0 and ω is a set of cycles and doubled edges of G; the function h is constant on the complementary components of these cycles and changes by 1 mod 2 across each cycle. We can think of the height modulo 2 as taking two values, or spins, on the faces of G, and the dimer chains are the spin-domain boundaries. In particular dimers on a nonbipartite graph model can in this way model the Ising model on an associated dual planar graph.

3 Thermodynamic limit

By *periodic planar graph* we mean a graph G, with energy function on edges, for which translations by elements of $\mathbb{Z}^2$ or some other rank-2 lattice $\Gamma \subset \mathbb{R}^2$ are isomorphisms of G preserving the edge energies, and such that the quotient $G/\mathbb{Z}^2$ is a finite graph. Without loss of generality we can take $\Gamma = \mathbb{Z}^2$. The standard example is $G = \mathbb{Z}^2$ with $\mathcal{E} \equiv 0$, which we refer to as "dimers on the grid". However other examples display different global behaviors and so it is worthwhile to remain in this generality.

For a periodic planar graph G, an ergodic probability measure on $\mathcal{M}(G)$ is one which is translation-invariant (the measure of a set is the same as any $\mathbb{Z}^2$-translate of that set) and whose invariant subsets have measure 0 or 1.

We will be interested in probability measures which are both ergodic and Gibbs (we refer to them as ergodic Gibbs measures, dropping the term "probability"). When G is bipartite, there are multiple ergodic Gibbs measures, see Theorem 3, below. When G is nonbipartite, it is conjectured that there is a single ergodic Gibbs measure.

In the remainder of this section we assume G is bipartite, and assume also that the $\mathbb{Z}^2$-action preserves the coloring of the edges as black and white (simply pass to an index-2 sublattice if not).

For integer $n > 0$ let $G_n = G/n\mathbb{Z}^2$, a finite graph on a torus (in other words, with periodic boundary conditions). For a dimer covering M of G_n, we define $(h_x, h_y) \in \mathbb{Z}^2$ to be the horizontal and vertical height change of M around the torus, that is, the net flux of $\omega(M) - \omega_0$ across a horizontal, respectively vertical, cut around the torus (in other words, h_x, h_y are the horizontal and vertical periods around the torus of the 1-form $\omega(M) - \omega_0$). The *characteristic polynomial* $P(z, w)$ of G is by definition

$$P(z, w) = \sum_{M \in \mathcal{M}(G_1)} e^{-\mathcal{E}(M)} z^{h_x} w^{h_y} (-1)^{h_x h_y}$$

here the sum is over dimer coverings M of $G_1 = G/\mathbb{Z}^2$, and h_x, h_y depend on M. The polynomial P depends on the base point ω_0 only by a multiplicative factor involving a power of z and w. From this polynomial most of the large-scale behavior of the ergodic Gibbs measures can be extracted.

The Gibbs measure on G_n converges as $n \to \infty$ to the (unique) ergodic Gibbs measure μ with smallest free energy $F = -\log Z$. The unicity of this measure follows from the strict concavity of the free energy of ergodic Gibbs measures as a function of the slope, see below. The free energy F of the minimal free energy measure is

$$F = -\frac{1}{(2\pi i)^2} \int_{S^1 \times S^1} \log P(z, w) \frac{dz}{z} \frac{dw}{w}$$

that is, minus the *Mahler measure* of P.

For any translation invariant measure ν on $\mathcal{M}(G)$, the average slope (s, t) of the height function for ν-almost every tiling is by definition the expected horizontal and vertical height change over one fundamental domain, that is, $s = \mathbb{E}[h(f + (1, 0)) - h(f)]$ and $t = \mathbb{E}[h(f + (0, 1)) - h(f)]$ where f is any face. This quantity (s, t) lies in the Newton polygon of $P(z, w)$ (the convex hull in $\mathbb{R}^2$ of the set of exponents of monomials of P). In fact, the points in the Newton polygon are in bijection with the ergodic Gibbs measures on $\mathcal{M}(G)$:

Theorem 3. *When G is a periodic bipartite planar graph, any ergodic Gibbs measure has average slope (s, t) lying in $N(P)$. Moreover, for every point $(s, t) \in N(P)$ there is a unique ergodic Gibbs measure $\mu(s, t)$ with that average slope.*

In particular this gives a complete description of the set of all ergodic Gibbs measures. The ergodic Gibbs measure $\mu(s, t)$ of slope (s, t) can be obtained as the limit of the Gibbs measures on G_n, when one conditions the configurations to have a particular slope approximating (s, t).

3.1 Ronkin function and surface tension

The *Ronkin function* of P is a map $R\colon \mathbb{R}^2 \to \mathbb{R}$ defined for $(B_x, B_y) \in \mathbb{R}^2$ by

$$R(B_x, B_y) = \frac{1}{(2\pi i)^2} \int_{S^1 \times S^1} \log P(ze^{B_x}, we^{B_y}) \frac{dz}{z} \frac{dw}{w}.$$

The Ronkin function is convex and its graph is piecewise linear on the complement of the *amoeba* $\mathbb{A}(P)$ of P, which is the image of the zero set $\{(z, w) \in \mathbb{C}^2 \mid P(z, w) = 0\}$ under the map $(z, w) \mapsto (\log |z|, \log |w|)$. See Figures 3 and 4 for an example.

The free energy $F(\mu(s, t))$ of $\mu(s, t)$, as a function of $(s, t) \in N(P)$, turns out to be the Legendre dual of minus the Ronkin function of $P(z, w)$: we have

$$F(\mu(s, t)) = -R(B_x, B_y) + sB_x + tB_y$$

where

$$s = \frac{\partial R(B_x, B_y)}{\partial B_x}$$
$$t = \frac{\partial R(B_x, B_y)}{\partial B_y}.$$

The continuous map $\nabla R\colon \mathbb{R}^2 \to N(P)$ which takes (B_x, B_y) to (s, t) is injective on the interior of $\mathbb{A}(P)$, collapses each bounded complementary component of $\mathbb{A}(P)$ to an integer point in the interior of $N(P)$, and collapses each unbounded complementary component of $\mathbb{A}(P)$ to an integer point on the boundary of $N(P)$.

Under the Legendre duality, the facets in the graph of the Ronkin function (that is, maximal regions on which R is linear) give points of non-differentiability of the free energy F, as defined on $N(P)$. We refer to these points of non-differentiability as "cusps". Cusps occur only at integer slopes (s, t). See Figure 5 for the free energy associated to the Ronkin function in Figure 4.

By Theorem 3, the coordinates (B_x, B_y) can also be used to parametrize the set of Gibbs measures $\mu(s, t)$ (but only those with slope (s, t) in the interior of $N(P)$ or on the corners of $N(P)$ and boundary integer points). This parametrization is not one-to-one since when (B_x, B_y) varies in a complementary

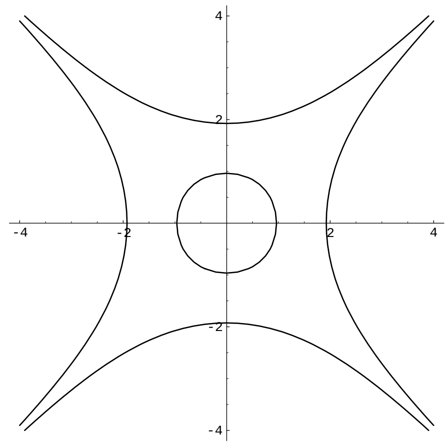

Figure 3: The amoeba of $P(z,w) = 5 + z + 1/z + w + 1/w$, which is the characteristic polynomial for dimers on the periodic "square-octagon" lattice.

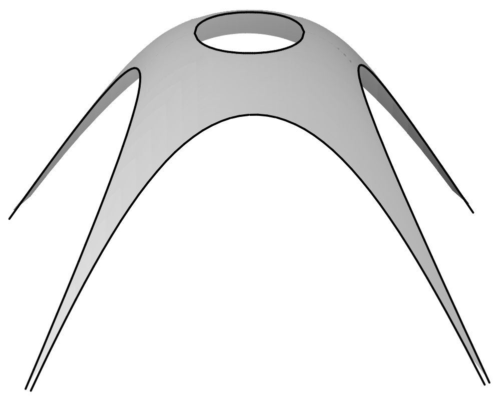

Figure 4: Minus the Ronkin function of $P(z,w) = 5 + z + 1/z + w + 1/w$.

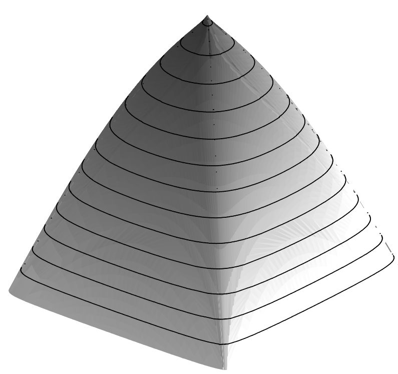

Figure 5: (Negative of) the free energy for dimers on the square-octagon lattice.

component of the amoeba, the measure $\mu(s,t)$ does not change. On the interior of the amoeba the parametrization is one-to-one.

The remaining Gibbs measures, whose slopes are on the boundary of $N(P)$, can be obtained by taking limits of (B_x, B_y) along the "tentacles" of the amoeba.

3.2 Phases

The Gibbs measures $\mu(s,t)$ can be partitioned into three classes, or phases, according to the behavior of the fluctuations of the height function. If we measure the height at two distant points x_1 and x_2 in G, the average height difference, $\mathbb{E}[h(x_1)-h(x_2)]$, is a linear function of x_1-x_2 determined by the average slope of the measure. The *height fluctuation* is defined to be the random variable $h(x_1)-h(x_2)-\mathbb{E}[h(x_1)-h(x_2)]$. This random variable depends on the two points and we are interested in its behavior when x_1 and x_2 are far apart.

We say $\mu(s,t)$ is

1. **Frozen** if the height fluctuations are bounded almost surely.
2. **Rough** (or **liquid**) if the covariance in the height function $\mathbb{E}[h(x_1)h(x_2)] - \mathbb{E}[h(x_1)]\mathbb{E}[h(x_2)]$ is unbounded as $|x_1 - x_2| \to \infty$.
3. **Smooth** (or **gaseous**) if the covariance of the height function is bounded but the height fluctuations are unbounded.

The height fluctuations can be related to the decay of the entries of K^{-1}, which are in turn related to the decay of the Fourier coefficients of $1/P$. In particular we have

Theorem 4. *The measure $\mu(s,t)$ is respectively frozen, rough, or smooth according to whether $(B_x, B_y) = (\nabla R)^{-1}(s,t)$ is in the closure of an unbounded complementary component of $\mathbb{A}(p)$, in the interior of $\mathbb{A}(P)$, or in the closure of a bounded component of $\mathbb{A}(P)$.*

The characteristic polynomials P which occur in the dimer model are not arbitrary: their algebraic curves $\{P = 0\}$ are all of a special type known as *Harnack curves*, which are characterized by the fact that the map from the zero-set of P in $\mathbb{C}^2$ to its amoeba in $\mathbb{R}^2$ is at most 2-to-1. In fact:

Theorem 5. *By varying the edge energies all Harnack curves can be obtained as the characteristic polynomial of a planar dimer model.*

3.3 Local statistics

In the thermodynamic limit (on a periodic planar graph), local statistics of dimer coverings for the Gibbs measure of minimal free energy can be obtained from the limit of the inverse of the Kasteleyn matrix on the finite toroidal graphs G_n. This in turn can be computed from the Fourier coefficients of $1/P$.

As an example, let G be the square grid $\mathbb{Z}^2$ and take $\mathcal{E} = 0$ (which corresponds to the uniform measure on configurations for finite graphs). An appropriate choice of signs for the Kasteleyn matrix is to put weights $1, -1$ on alternate horizontal edges and $i, -i$ on alternate vertical edges in such a way that around each white vertex the weights are cyclically $1, i, -1, -i$. For this choice of signs we have

$$K^{-1}_{(0,0),(x,y)} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{-i(\theta x + \phi y)} d\theta d\phi}{2\sin\theta + 2i\sin\phi}.$$

This integral can be evaluated explicitly, see Figure 6 for values of $K^{-1}_{(0,0),(x,y)}$ for (x,y) near the origin (note that by translation invariance, $K^{-1}_{(x',y'),(x,y)} = K^{-1}_{(0,0),(x-x',y-y')}$). Values in other quadrants can be obtained by $K^{-1}_{(0,0),(x,y)} = -iK^{-1}_{(0,0),(-y,x)}$.

As a sample computation, using Theorem 2, the probability that the dimer covering the origin points to the right and, simultaneously, the one covering $(0,1)$ points upwards is

$$K_{(0,0),(1,0)}K_{(0,2),(0,1)} \det \begin{pmatrix} K^{-1}_{(0,0),(1,0)} & K^{-1}_{(0,0),(0,1)} \\ K^{-1}_{(0,2),(1,0)} & K^{-1}_{(0,2),(0,1)} \end{pmatrix} = 1 \cdot (-i) \cdot \det \begin{pmatrix} \frac{1}{4} & \frac{-i}{4} \\ -\frac{1}{4} + \frac{1}{\pi} & \frac{i}{4} \end{pmatrix} = \frac{1}{4\pi}.$$

Another computation which follows is the decay of the edge covariances. If e_1, e_2 are two edges at distance d, then $\Pr(e_1 \& e_2) - \Pr(e_1)\Pr(e_2)$ decays quadratically in $1/d$, since $K^{-1}((0,0),(x,y))$ decays like $1/(|x| + |y|)$.

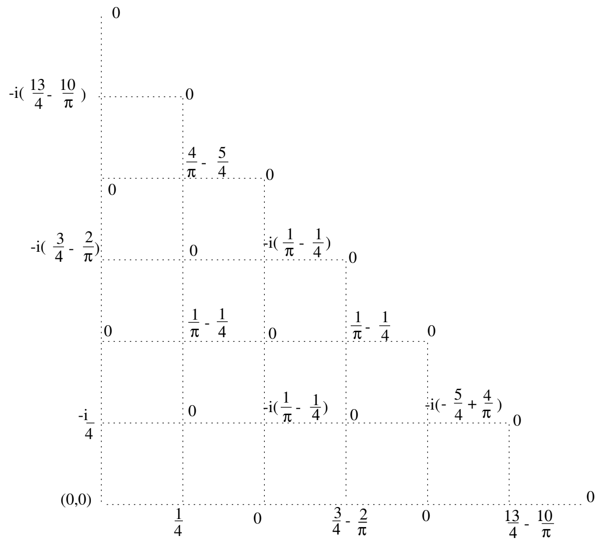

Figure 6: Values of K^{-1} on $\mathbb{Z}^2$ with zero energies.

4 Scaling limits

The scaling limit of the dimer model is the limit when the lattice spacing tends to zero.

Let us define the scaling limit in the following way. Let $\epsilon\mathbb{Z}^2$ be the square grid scaled by ϵ, so the lattice mesh size is ϵ. Fix a Jordan domain $U \subset \mathbb{R}^2$ and consider for each ϵ a subgraph U_ϵ of $\epsilon\mathbb{Z}^2$, bounded by a simple polygon, which tends to U as $\epsilon \to 0$. We are interested in limiting properties of random dimer coverings of U_ϵ, in the limit as $\epsilon \to 0$, for example the fluctuations of the height function and edge densities.

The limit depends on the (sequence of) boundary conditions, that is, on the exact choice of approximating regions U_ϵ. By changing U_ϵ one can change the limiting rescaled height function along the boundary. It is conjectured that the limit of the height function along the boundary of U_ϵ (scaled by ϵ...and assuming this limit exists) determines essentially all of the limiting behavior in the interior, in particular the limiting local statistics.

Therefore, let u be a real-valued continuous function on the boundary of U. Consider a sequence of subgraphs U_ϵ of $\epsilon\mathbb{Z}^2$, as $\epsilon \to 0$ as above, and whose height function along the boundary, when scaled by ϵ, is approximating u. We discuss the limit of the model in this setting.

4.1 Crystalline surfaces

The height function allows us to view dimer coverings as random surfaces in $\mathbb{R}^3$: to a dimer covering of G, one associates the graph of its height function, extended in a piecewise linear fashion over the edges and faces of the dual G^*. These surfaces are then piecewise linear random surfaces, which resemble crystal surfaces in the sense that microscopically (on the scale of the lattice) they are rough, whereas their long-range behavior is smooth and facetted, as we now describe.

In the scaling limit, boundary conditions as described in the last paragraph of the previous section are referred to as "wire-frame" boundary conditions, since the graph of the height function can be thought of as a (random) surface spanning the wire frame defined by its boundary values.

In the scaling limit, there is a law of large numbers which says that the Gibbs measure on random surfaces (which is unique since we are dealing with a finite graph) concentrates, for fixed wire-frame boundary conditions, on a single surface S_0. That is, as the lattice spacing ϵ tends to zero, with probability tending to 1 the random surface lies close to a limiting surface S_0. The surface S_0 is the unique surface which minimizes the total *surface tension*, or *free energy*, for its fixed boundary values, that is, minimizes the integral over the surface of the $F(\mu(s,t))$ where (s,t) is the slope of the surface at the point being integrated over. Existence and unicity of the minimizer follow from the strict convexity of the free energy/surface tension as a function of the slope.

At a point where the free energy has a cusp, the crystal surface S_0 will in general have a facet, that is, a region on which it is linear. Outside of the facets, one expects that S_0 is analytic, since the free energy is analytic outside the cusps.

4.2 Fluctuations

While the scaled height function ϵh in the scaling limit converges to its mean value h_0 (whose graph is the surface S_0), the fluctuations of the unrescaled the height function $h - \frac{1}{\epsilon}h_0$ will converge in law to a random process on U.

In the simplest setting, that of honeycomb dimers with $\mathcal{E} \equiv 0$, and in the absence of facets, the height fluctuations converge to a continuous Gaussian process, the image of the Gaussian free field on the unit disk $\mathbb{D}$ under a certain diffeomorphism Φ (depending on h_0) of $\mathbb{D}$ to U.

In the particular case $h_0 = 0$, Φ is the Riemann map from $\mathbb{D}$ to U and the law of the height fluctuations is just the Gaussian free field on U (defined to be the Gaussian process whose covariance kernel is the Dirichlet Green's function). The conformal invariance of the Gaussian free field is the basis for a number of conformal invariance properties of the honeycomb dimer model.

4.3 Densities of motifs

Another observable of interest is the density field of a motif. A motif is a finite collection of edges, taken up to translation. For example consider, for the square grid, the 'L' motif consisting of a horizontal domino and a vertical domino aligned to form an ell, which we showed above to have a density $1/4\pi$ in the thermodynamic limit. The probability of seeing this motif at any given place is $1/4\pi$. However in the scaling limit one can ask about the fluctuations of the occurrences of this motif: in a large ball around a point x, what is the distribution of $N_L - A/4\pi$, where N_L is the number of occurrences of the motif, and A is the area of the ball? These fluctuations form a random field, since there is a long-range correlation between occurrences of the motif.

It is known that on $\mathbb{Z}^2$, for the minimal free energy ergodic Gibbs measure, the rescaled density field

$$\frac{1}{\sqrt{A}}\left(N_L - \frac{A}{4\pi}\right)$$

converges as $\epsilon \to 0$ weakly to a Gaussian random field which is a linear combination of a directional derivative of the Gaussian free field and an independent white noise. A similar result holds for other motifs.

The joint distribution of densities of several motifs can be non-Gaussian and is unknown at present.

5 Further Reading

- Cohn H., Kenyon R., Propp J. (2001), A variational principle for domino tilings, J. Amer. Math. Soc., **14**, no.2, 297-346.
- Kasteleyn P. (1967), *Graph theory and crystal physics*, Graph Theory and Theoretical Physics, pp. 43–110 Academic Press, London.
- Kenyon R., Okounkov A. , Sheffield S., Dimers and Amoebae, *Ann. Math. 2005. math-ph/0311005.*