



# a Dimer?

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A *dimer* is a polymer with only two atoms. A dimer covering of a graph  $G$  is a collection of edges that covers all the vertices exactly once, that is, each vertex is the endpoint of a unique edge. One can think of vertices of  $G$  as univalent atoms, each bonding to exactly one neighbor. Dimer coverings are also called perfect matchings. The *dimer model* is the study of natural measures (“Gibbs measures”) on the set of dimer coverings of a graph, usually a periodic planar graph such as  $\mathbb{Z}^2$ . While not a very realistic physical model, the dimer model has intrinsic interest as an exactly solvable model that exhibits certain types of phase transitions. Since phase transitions are quite complicated phenomena in nature, any approximate model whose phase transitions can be studied analytically is valuable.

Kasteleyn, contemporaneously with Temperley and Fisher, showed how to count the number of dimer coverings of an  $m \times n$  square grid, and later on any planar graph. While Kasteleyn’s result holds for any planar graph, the statement is particularly simple when  $G$  is a subgraph of the honeycomb graph  $H$  (the graph of the regular tiling of the plane by hexagons) bounded by a simple polygon (a “simply connected” subgraph). Then the number of coverings  $Z$  is the square root of the determinant of the adjacency matrix of  $G$ . That is,

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$Z = \sqrt{\det K}$  where  $K$  is the matrix indexed by the vertices of  $G$  defined by  $K_{v,v'} = 1$  or 0 according to whether  $v, v'$  are adjacent or not. A similar statement (but with extra signs in  $K$ ) holds for any planar graph. One can compute, for example, the number of dimer coverings of an  $m \times n$  grid to be

$$\prod_{\ell=1}^m \prod_{k=1}^n \left| 2 \cos \frac{\pi \ell}{m+1} + 2i \cos \frac{\pi k}{n+1} \right|^{1/2}.$$

For example, the number of domino tilings of a checkerboard (which are just dimers on an  $8 \times 8$  grid) is  $12988816 = 2^4 17^2 53^2$  tilings.

Dimer coverings of bipartite graphs such as the honeycomb graph or square grid can be viewed as (random) surfaces in  $\mathbb{R}^3$ . Concretely, dimer coverings of the honeycomb or any of its simply connected subgraphs can be represented as tilings of the plane with  $60^\circ$  rhombi: in this case each atom is a triangle, and dimers are obtained by gluing adjacent triangles along an edge. See Figure 1 for an example. Clearly any such rhombus tiling can be viewed as the projection of a three-dimensional piecewise-linear surface. The function on a tiling that gives the third coordinate of this projection is called the *height function*. A similar kind of height function also exists for domino tilings, or dimers on any bipartite planar graph, though the definition is a little more complicated.

The tiling in Figure 1 is perfectly random, and yet one can see that there is something regular about its height function. In fact, as we take tilings of a fixed region (e.g., a polygon) with smaller and smaller rhombi, the height functions of a typical

tiling will converge to some nonrandom surface called the *limit shape*. What is this limit shape? Figure 1 shows it has some linear pieces—those occur in the *frozen* regions near the corners where all tiles there are lined up the same way. How large are the fluctuations around the limit shape? What is the probability of occurrence of some fixed local pattern at a given point?

The answers to these and other questions depend on the asymptotics of the inverse Kasteleyn matrix  $K^{-1}$  as the size of the graph goes to infinity. Since  $K$  can be viewed (using work of the first author) as a discretization of the  $\bar{\partial}$ -operator, it is not so surprising that complex analysis and algebraic geometry enter the scene. In the nonfrozen region, the limit shape satisfies a nonlinear elliptic PDE closely related to the complex Burgers equation  $\phi_x + \phi\phi_y = 0$ . To be precise, the arguments of  $\phi$ ,  $1 - 1/\phi$  and  $1/(1 - \phi)$  are proportional to the densities of tiles of the three orientations. These densities extend continuously to constants on the frozen regions. For polygonal domains like the one in Figure 1, the solution to this free boundary problem is algebraic. In particular, it turns out that the boundary of the frozen region in the figure is an inscribed cardioid (a degree 4, genus 0 curve with 1 real and 2 complex cusps—the yellow curve in the figure).

From the complex Burgers equation, the nonfrozen part of the limit shape comes equipped with a natural complex and, hence, conformal structure, which is different from the conformal structure induced from the plane. It is expected that in this new conformal structure the fluctuations around the limit shape are described by a conformally invariant process called the *Gaussian free field*, a two-dimensional cousin of Brownian motion. It is still an open problem (even for the honeycomb graph) to prove this in complete generality. Partial results are available in recent work by the authors.

Similarly, it is expected that the limiting probability of observing any given finite local pattern of tiles at a given point is a function that depends only on the slope of the limit shape at that point. That is, if two limit shapes have the same slopes at corresponding points, then they have the same local measures (defined by the densities of local patterns) there. All these probabilities can be computed explicitly. For example, if  $s$  is the  $y$ -slope of the height function, then the density of horizontal rhombi is  $s + 1/3$ , and the covariance between two horizontal rhombi, one of which is  $k$  steps directly above the other, equals  $-\frac{\sin^2 s\pi k}{(\pi k)^2}$ . One derives nearly identical formulas for correlations of eigenvalues of random matrices. This is one indication that dimers and random matrices have very much in common. Indeed, in a certain natural sense a

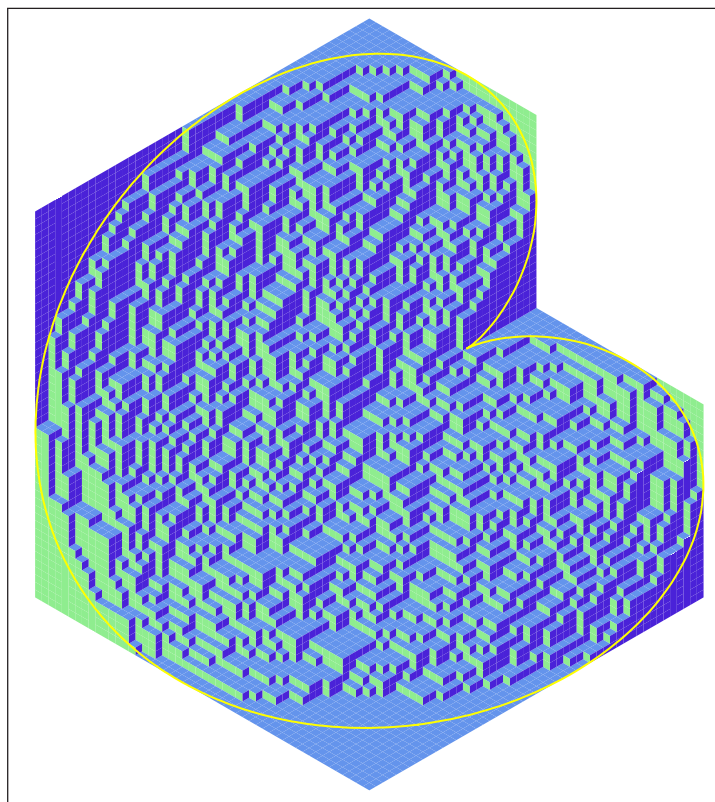


Figure 1.

random dimer covering is a discretization of the time evolution of the eigenvalues of a random unitary matrix.

Instead of the honeycomb graph, one can make similar calculations for dimers on any periodic bipartite planar graph. In fact, it is natural to allow the edges of the graph to be weighted, in which case the weight of a dimer covering is proportional to the product of the weights of its edges. Again the statistics can be worked out exactly using Kasteleyn's method, and one obtains more complicated PDEs for the limit shapes. Curiously, these PDEs can all be solved, and one can find the limit shapes via analytic data, in a way resembling Weierstrass's parametrization of minimal surfaces in  $\mathbb{R}^3$  via analytic data. These more general models enjoy a further property that their limit shapes can have facets in many other rational directions. The interplay between analyticity and facet formation makes for a fascinating study in real algebraic geometry.

#### Further Reading

- [1] P. KASTELEYN, Graph theory and crystal physics, *Graph Theory and Theoretical Physics*, Academic Press, London, 1967, pp. 43–110.
- [2] H. COHN, R. KENYON, J. PROPP, A variational principle for domino tilings. *J. Amer. Math. Soc.* **14** (2001), no. 2, 297–346.
- [3] R. KENYON, The planar dimer model with boundary: A survey, *Directions in mathematical quasicrystals*, CRM Monogr. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2000, pp. 307–328.