INTEGRABLE SYSTEMS IN THE DIMER MODEL

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1. Convex integer polygon

2. Minimal bipartite graph on $\mathbb{T}^2$

3. Cluster integrable system.
Line bundle on graph = edge weights modulo gauge = monodromies around faces \((w_i)\) and homology generators of torus \((z_1, z_2)\) subject to one condition: \(\prod w_i = 1.\)
Define a Poisson structure on the moduli space of line bundles by the formula

\[ \{w_i, w_j\} = \varepsilon_{ij} w_i w_j \]  

(extend using Leibniz rule)

where \( \varepsilon \) is a skew-symmetric form

\[ \varepsilon_{ij} = 1 \text{ if } w_i \text{ and } w_j \]  

\[ \varepsilon_{ij} = -1 \text{ if } \]  

\[ \varepsilon_{ij} = 0 \text{ else.} \]

A similar rule for \( \{w_i, z_j\} \) and \( \{z_i, z_j\} \).

\( \varepsilon \) is the intersection form on cycles on the “conjugate” surface obtained by reversing the cyclic orientation at black vertices.
\[ w_1 \begin{pmatrix} 0 & 1 & 2 & -2 & -1 & 1 & -2 \\ -1 & 0 & 1 & 2 & -2 & 0 & 0 \\ -2 & -1 & 0 & 1 & 2 & -2 & 4 \\ 2 & -2 & -1 & 0 & 1 & 0 & 0 \\ 1 & 2 & -2 & -1 & 0 & 1 & -2 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 \\ 2 & 0 & -4 & 0 & 2 & 0 & 0 \end{pmatrix} \]

\[ \varepsilon = w_4 \]
Goal: define commuting Hamiltonians.

\[ H_1 = z_1^{-1}(1 + w_1 + w_1 w_2 + w_1 w_2 w_3 + w_1 w_2 w_3 w_4) \]
\[ H_2 = z_1^{-1}(w_1^2 w_2 w_3 + w_1^2 w_2^2 w_3 + w_1^2 w_2 w_3 w_4 + w_1^2 w_2^2 w_3 w_4 + w_1^2 w_2^2 w_3 w_4) \]

\[ C_1 = z_1^2 z_2 \]
\[ C_2 = w_1^4 w_2^3 w_3^2 w_4 z_1 z_2^3 \]

Casimirs (commute with everything)
A “zig-zag” path is in the kernel of $\varepsilon$ (and these generate the kernel). “Casimirs”
The Hamiltonians are normalized sums of weighted dimer covers.

A dimer cover has a weight = product of edge weights.

Fix a “base point” dimer cover.
Combining with another cover gives a set of cycles.
The ratio of weights is the product of the cycle monodromies.
(and so is independent of gauge)
Let $M(G)$ be the set of dimer covers of $G$.

Define the partition function

$$P(z_1, z_2) = \sum_{\text{dimer covers } m} \nu(m) z_1^i z_2^j (-1)^{ij}$$

The normalized coefficients of $P(z_1, z_2)$ are the Hamiltonians.

$$(\text{divide by weight of a zig-zag path})$$

$$H_{i,j} = z_1^i z_2^j \sum_{m \in \Omega_{i,j}} \nu(m)$$
Coefficients of the dimer partition function

\[ C_{0,0} = 1 + w_1 + w_1 w_2 + w_1 w_2 w_3 + w_1 w_2 w_3 w_4 \]
\[ C_{0,1} = w_1^2 w_2 w_3 + w_1^2 w_2^2 w_3 + w_1^2 w_2 w_3 w_4 + w_1^3 w_2^2 w_3 w_4 + w_1^3 w_2^2 w_3^2 w_4 \]
Proof of commutativity of Hamiltonians.

ε is a sum of local contributions at vertices:

\[ \varepsilon_{R,B}(v) = \frac{1}{2} \quad \varepsilon_{R,B}(v) = 0 \quad \varepsilon_{R,B}(v) = 1 \quad \varepsilon_{R,B}(v) = 0 \]

and reverse sign if reverse vertex color or any path orientation.

\[ \varepsilon_{R,B} = 0 \]
For a pair of dimer covers $R, B$, let $R^*, B^*$ be obtained by reversing colors on all topologically trivial loops.

\[
\{R, B\} + \{R^*, B^*\} = \varepsilon_{R,B} RB + \varepsilon_{R^*,B^*} RB = (\varepsilon_{R,B} + \varepsilon_{R^*,B^*}) RB = 0
\]

also use

**Lemma:** topologically nontrivial loops give net contribution zero.
1. Convex integer polygon $N$
   \[\downarrow\text{triple crossing diagram}\]
2. Minimal bipartite graph on $\mathbb{T}^2$
   \[\downarrow\text{line bundles}\]
3. Cluster integrable system.
Theorem [Goncharov-K]
This Poisson bracket defines a completely integrable system of dimension \(2 + 2\text{Area}(N)\), with symplectic leaves of dimension \(2\text{int}(N)\), (twice the number of interior vertices). A basis for the Casimir elements is given by (ratios of) boundary coefficients of \(P\). The commuting Hamiltonians are the normalized interior coefficients of \(P\).

A quantum integrable system can be defined using \(q\)-commuting variables:

\[
    w_i w_j = q^{2\epsilon_{ij}} w_j w_i.
\]
\[ \frac{\partial z_i}{\partial t} = \{z_i, H\} \]
\[ \frac{\partial w_i}{\partial t} = \{w_i, H\} \]
Start: a convex polygon with vertices in $\mathbb{Z}^2$. 
Geodesics on the torus, one for each primitive edge of $N$. 
Isotope to a “triple-crossing diagram” [D. Thurston]
Lemma: $|\text{white vertices}| = |\text{black vertices}| = |\text{faces}| = 2\text{Area}(N)$.

Obtain a bipartite graph
Modding out by Casimirs, there is a change of variables
\[ \{w_i\} \leftrightarrow \{p_1, \ldots, p_k, q_1, \ldots, q_k\} \]

changing the symplectic form to the standard one
\[ \sum_{i=1}^{k} \frac{dp_i}{p_i} \wedge \frac{dq_i}{q_i} \]

it has the form:
\[ w_i = \frac{\det(A_1) \det(A_3)}{\det(A_2) \det(A_4)} \]

where the $A_i$ are “generalized Vandermonde” matrices in $p_j, q_j$. 
Consider for example the following graph
first define "A" variables

\[ A_{i,j} = \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ p_1q_1 & p_2q_2 & p_3q_3 & p_4q_4 \end{pmatrix} \]

"generalized vandermonde"
the $w$ variables (monodromies) are cross ratios of these:

$$w_i = \frac{\det(A_1) \det(A_3)}{\det(A_2) \det(A_4)}$$
\[ w_i = \frac{\det \begin{pmatrix} p_1^2 q_1 & p_2^2 q_2 & p_3^2 q_3 & p_4^2 q_4 \\ p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ p_1 q_1 & p_2 q_2 & p_3 q_3 & p_4 q_4 \end{pmatrix}}{\det \begin{pmatrix} p_1^2 q_1 & p_2^2 q_2 & p_3^2 q_3 & p_4^2 q_4 \\ p_1 & p_2 & p_3 & p_4 \\ p_1^2 & p_2^2 & p_3^2 & p_4^2 \\ p_1 q_1 & p_2 q_2 & p_3 q_3 & p_4 q_4 \end{pmatrix}} \cdot \det \begin{pmatrix} 1 & 1 & 1 & 1 \\ p_1 & p_2 & p_3 & p_4 \\ 1 & p_2 & p_3^2 & p_4^2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \]
\[ \frac{\partial z_i}{\partial t} = \{z_i, H\} \]

\[ \frac{\partial w_i}{\partial t} = \{w_i, H\} \]
The map intertwines the action of the torus $T$ on $X$ with the action provided by the action of $T$ on the surface $N$.

In Section 7 we prove a weaker statement: $S$ is a finite cover over the generic part of $S$. It implies the independence of the Hamiltonians.

### 1.5 Analogies between dimers, Teichmüller theory and cluster varieties.

Cluster Poisson varieties provide a framework for study of both (classical and higher) Teichmüller theory [FG] and theory of dimers. Here is a dictionary relating key objects in these three theories.

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What distinguishes these two examples – the dimer theory and the Teichmüller theory – from the general theory of cluster Poisson varieties is that in each of them the set of real / tropical real points of the relevant cluster variety has a meaningful and non-trivial interpretation as the moduli space of some geometric objects.

Here by moduli space of certain objects related to the toric surface $N$ we mean the space parametrizing the orbits of the torus $T$ acting on the objects. For example, the
Triangle flip

\[
\begin{align*}
\lambda_0' &= \lambda_0^{-1} \\
\lambda_1' &= \lambda_1 (1 + \lambda_0) \\
\lambda_2' &= \lambda_2 (1 + \lambda_0^{-1})^{-1} \\
\lambda_3' &= \lambda_3 (1 + \lambda_0) \\
\lambda_4' &= \lambda_4 (1 + \lambda_0^{-1})^{-1}
\end{align*}
\]

Urban renewal

\[
\begin{align*}
w_0' &= w_0^{-1} \\
w_1' &= w_1 (1 + w_0) \\
w_2' &= w_2 (1 + w_0^{-1})^{-1} \\
w_3' &= w_3 (1 + w_0) \\
w_4' &= w_4 (1 + w_0^{-1})^{-1}
\end{align*}
\]