DISCRETE ANALYTIC FUNCTIONS AND INTEGRABILITY

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based on work with Scott Sheffield (MIT) and Alexander Goncharov (Yale)
Menelaus Theorem

\[
\frac{ad}{bc} = \frac{eh}{fg}
\]
A dynamical system
draw in diagonals...
Tilings and networks

○ = variables
● = equations

\[ F - E + V = 1 \]
\[ ○ - (● + 3) + 4 = 1 \]

Nonsingular?
put complex weights on edges
put complex weights on edges

\[
\frac{ad}{bc} = \frac{eh}{fg} < 0
\]
put complex weights on edges
This weighted adjacency matrix is called the *Kasteleyn matrix* \( K_{BW} \) (on each face, the alternating product of weights has sign \((-1)^{\ell/2+1}\)).

**Thm (Kasteleyn, 1965)**

\[
|\det K_{BW}| = \sum_{\text{dimer covers } m} \prod_{e \in m} w_e.
\]

Proof idea:

\[
\det K_{BW} = \sum_{\sigma \in S_n} (-1)^\sigma K_{b_1 w_{\sigma(1)}} \cdots K_{b_n w_{\sigma(n)}}
\]

nonzero terms all have the same sign. \(\square\)
dimer cover
Def. A function $f \in \mathbb{C}^B$ is \textit{discrete analytic} if $Kf = 0$.

Example: $\mathbb{Z}^2$

\[
Kf(z) = if(z + 1) - if(z - 1) - f(z + i) + f(z - i)
= i(f(z + 1) - f(z - 1) + if(z + i) - if(z - i))
= i(\partial_x + i\partial_y)f(z)
\]
Thm: A convex polygon $P$ can be tiled by rational (homothety of one with rational vertices) polygons iff $P$ is rational.
Thm: [Dylan Thurston] A tiling of a convex polygon can be reduced to the trivial tiling using Menelaus moves and "diagonal" moves.

Diagonal move: (add/remove diagonal from a convex tile)
Back to the dynamical system:

The composition \( \tau_{\text{odd}} \circ \tau_{\text{even}} \) is an integral dynamical system, the HBDE (octahedron recurrence.)

More generally, we get a multidimensional version thereof.
Another (related?) integrable system:

Map $\mathbb{Z}^2 \to \mathbb{R}^2$ so that faces are circular:
THANK YOU