3D Young diagram (plane partition).

The number of diagrams with volume $n$ has generating function

$$M(q) = \prod_{n=1}^{\infty} (1 - q^n)^{-n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + \ldots$$
Open question: What are the generalized (weighted) versions?

Example: \[ P = 1 + z + w - zw \]

\[ Z(q) = \frac{M(q)M(q^4)^2}{M(q^2)} \quad \text{Domino Macmahon function} \]

\[ = 1 + q + 2q^2 + 5q^3 + 10q^4 + 18q^5 + \ldots \]
With a “roof” of $n$ balls:

$$Z(q) = \frac{M(q)M(q^4)^2}{M(q^2)}(1 + q)^{n-1}(1 + q^3)^{n-2} \ldots (1 + q^{2n-3})^1.$$
Generalized MacMahon function

Define: \[ M(z, q)^{-1} = (1 - zq)(1 - zq^2)^2(1 - zq^3)^3 \ldots \]

with \(1 \times m\) periodic weights on horizontal edges

\[ Z(q) = \prod_{1 \leq i, j \leq m} M\left( \frac{a_iq^i}{a_jq^j}, q^m \right). \]
Conjecture: $Z(q)$ has a product form iff $P$ has genus zero.

Question: Compute the total asymptotic surface tension of these solutions, as a function of $Q$. 