Math 0540 extra problem solutions

Page 279, #7.

The matrix has characteristic polynomial $z^2-4z-5 = (z-5)(z+1)$, which has distinct real roots. So it is diagonalizable. The eigenvalues are $\{-1, 5\}$ and corresponding eigenvectors are $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Therefore $A = Q \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} Q^{-1}$ where Page 322, #1 is the matrix of eigenvectors.

Thus

$$A^n = Q \begin{pmatrix} (-1)^n & 0 \\ 0 & 5^n \end{pmatrix} Q^{-1} = \frac{1}{3} \begin{pmatrix} 5^n + 2(-1)^n & -2(-1)^n + 2 \cdot 5^n \\ -1(-1)^n + 5^n & (-1)^n + 2 \cdot 5^n \end{pmatrix}$$

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Since $\dim(E_{\lambda_1}) = n - 1$ and $\dim(E_{\lambda_2}) \geq 1$ (the dimension of an eigenspace is always at least one), we must have $\dim(E_{\lambda_2}) = 1$, since bases for these spaces are independent. Thus A has a basis of eigenvectors: a basis for E_{λ_1} union a nonzero vector in E_{λ_2}. So A is diagonalizable.

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(a) False, $\{0\}$ is always invariant.
(b) True, proved in class (Thm 5.21 in the book)
(c) False, for example $w = 2v$ generates the same cyclic subspace as v.
(d) False, for example if $v \in N(T)$ and $v \neq 0$.
(e) True, the characteristic polynomial satisfies this.
(f) True, see problem 19 below.
(g) True, although we didn't talk about direct sum of matrices.

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(a) Yes. The derivative of a polynomial of degree at most two has degree at most two.
(b) No. $x^2 \in W$ but $T(x^2) = x^3 \notin W$.
(c) Yes. $T(t, t, t) = (3t, 3t, 3t) \in W$.
(d) Yes. $T(f(t)) = ct$ for some constant c which is in W for any f.
(e) No, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in W$ by $T(A) \notin W$.

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(a) $\{e_1, Te_1, T^2e_1\}$. One can see that $T^3e_1 = 3T^2e_1 - 3Te_1 - 3e_1$.

(b) $\{x^3, 6x\}$.

(c) $\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$.

(d) $\{z, T(z)\}$. We have $T^2(z) = 3T(z)$.

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If there is a polynomial g such that $g(T)v = w$, then w is a linear combination of $\{T^kv\}_{k=0,1,\dots}$ and so is in the cyclic subspace generated by v. Conversely, If w is in this cyclic subspace, then $w = \sum_{k=0}^{m} a_kT^kv$ for some m and so $w = g(T)v$ where $g(z) = \sum_{k=0}^{m} a_kz^k$.

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If $k = 1$ the result is true: the 1×1 matrix $(-a_0)$ has characteristic polynomial $-a0 - z$. Assuming it is true up to $k-1$, if we expand the determinant of $M_k - zI$ along row 1 we get

$$\begin{aligned} \det(M_k - zI) &= -z\det(M_{k-1} - zI) + (-1)^{k-1}(-a_0)\det(I) \\ &= (-1)^k a_0 - z(-1)^{k-1}(a_1 + a_2t + \cdots + a_{k-1}t^{k-2} + t^{k-1}). \end{aligned}$$

The result follows.