Math 1260
final exam solutions

1. Points $1+i, 3+2i$ are two vertices of a square. Find the two other vertices in all possible cases.

There are three cases. The difference $(3+2i)-(1+i)$ is an edge or a diagonal of the square. The other vertices are $1+i+(3+2i-(1+i))x$ where x is one of $i, 1+i, -i, -1-i, (1+i)/2, (1-i)/2$. This yields $3i, 2+4i, 2-i, 4, \frac{3}{2}+\frac{5}{2}i, \frac{5}{2}+\frac{i}{2}$.

2. Let $0<a<1$. Show that $\phi(z)=\frac{z-a}{1-az}$ maps the unit disk $\{z:|z|<1\}$ bijectively to itself.

Since ϕ has an inverse (obtained by solving $w=\frac{z-a}{1-az}$ for z), it is injective. We need to show that ϕ maps the unit disk both into and onto the unit disk. (TWO things to show!) One way is to show that it maps the unit circle to itself, and at least one point inside to a point inside (so that it doesn't switch the inside and outside). To show that it maps the circle to itself, just check that three points on the circle map to three points on the circle, since linear fractional transformations send circles to circles. ϕ sends 1 to 1, -1 to -1, and i to $\frac{i-a}{1-ia}$ which has modulus

$$\frac{\sqrt{1+a^2}}{\sqrt{1+a^2}}=1$$

since a is real. To see that ϕ maps the inside of the circle to the inside, note that $\phi(0)=a$ which is inside the circle by hypothesis.
An alternative proof is just to check the two things: if $|z|<1$, then $|\phi(z)|<1$, and if $|\phi(z)|<1$ then $|z|<1$.

3. Compute

$$\int_{|z|=4}\frac{z^3}{\sin z}dz.$$

Poles at $\pm\pi$, with a removable singularity at 0. The residues at the poles are $\mp\pi^3$, respectively, and so the integral is zero.

4. Show that, under an appropriate choice of branch for $\sqrt{z}$, the function $f(z)=\frac{\sin\sqrt{z}}{\sqrt{z}}$ is analytic on $\mathbb{C}\setminus\{0\}$ with an isolated singularity at 0. What kind of isolated singularity does it have? Compute the Laurent expansion about $z=0$.

Although neither $\sqrt{z}$ nor $\sin\sqrt{z}$ is analytic on $\mathbb{C}$, their ratio is analytic on $\mathbb{C}\setminus\{0\}$, as long as you use the same branch cut for both numerator and denominator. The easiest way to see this is via the power series expansion, which is

$$\frac{1}{\sqrt{z}}(\sqrt{z} - \frac{z^{3/2}}{3!} + \frac{z^{5/2}}{5!} - \dots) = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \dots.$$

This is the power series for $\sin\sqrt{z}/\sqrt{z}$, a priori only defined on $\mathbb{C} \setminus (-\infty, 0]$. This power series has infinite radius of convergence and so defines an analytic function on all of $\mathbb{C}$. The isolated singularity at the origin is removable, since there are no terms with negative exponent in the expansion.

5. Compute $\int_{|z|=4} \tan z\, dz$. Let $\tan z = f_0(z) + f_1(z)$ be the Laurent decomposition of $\tan z$ on the annulus $2 < |z| < 4$. What is $f_1(z)$?

$\tan z$ has two poles inside $|z| = 4$, at $\pm\frac{\pi}{2}$. The residues there are both -1. The integral is therefore $-4\pi i$. The function f_1 is the sum of the principal parts at the poles inside the circle $|z| = 4$, and is therefore

$$f_1(z) = -\frac{1}{z - \frac{\pi}{2}} - \frac{1}{z + \frac{\pi}{2}}.$$

6. For what values of $z \in \mathbb{C}$ is

$$\sum_{n=0}^{\infty} \left(\frac{1 - iz}{i - z}\right)^n$$

convergent? Absolutely convergent?

A geometric series is convergent when and only when it is absolutely convergent. So we need only check when $\left|\frac{1-iz}{i-z}\right| < 1$. This occurs when $\Im z < 0$. (you can see this by noting that $\frac{1-iz}{i-z}$ maps the lower half plane bijectively to the unit disk, since it maps $1 \to -1, -1 \to 1, 0 \to -i, -i \to 0$.

7. Show that the series $\sum_{n\in\mathbb{Z}} e^{-n^2+z/n}$ converges and represents an entire function of z.

We need to ignore the term $n = 0$ in the sum, my mistake. Let $M > 0$. If $|z| < M$ then $|e^{-n^2+z/n}| \leq e^{-n^2+M/n} \leq e^{-n^2+M}$. Therefore for $N > 0$,

$$|\sum_{|n|>N} e^{-n^2+z/n}| \leq e^M \sum_{|n|>N} e^{-n^2} \leq e^M \sum_{|n|>N} e^{-|n|} = \frac{2e^{M-N}}{1 - e^{-1}}$$

which tends to zero as $N \to \infty$. Thus the series converges, and converges **uniformly** when $|z| < M$ since the error does not depend on z. A uniformly convergent sum of analytic function is analytic, so the sum represents an analytic function in $\{z : |z| < M\}$. Since M was arbitrary, the sum is analytic in the whole plane.

8. Compute

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2 + x + 1} dx.$$

Write this as

$$\Re \int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2 + x + 1} dx.$$

Do the integral from $-N$ to N followed by a semicircular arc in the upper half place connecting the endpoints. The numerator is bounded by 1 on the semicircle (note that $\cos(2z)$ is NOT bounded by 1 on this semicircle!) and using the ML-estimate we find that the integral on the semicircle goes to zero with $N \to \infty$. The integral over the closed contour is $2\pi i$ times the residue at the unique pole inside, which is at $-\frac{1}{2} + \frac{\sqrt{3}}{2}$. The answer is

$$\Re\left(2\pi i e^{2i(-1/2+i\sqrt{3}/2)}/(i\sqrt{3})\right) = \frac{2\pi}{\sqrt{3}} e^{-\sqrt{3}} \cos 1.$$