Abstract

Following Rubinstein [Ru], Iwaniec-Luo-Sarnak [ILS], and Katz-Sarnak [KS1], [KS2], we use the 1- and 2-level densities to study the distribution of low lying zeros for families of elliptic curves.

For any automorphic cuspidal L-function, the n-level correlations of the high zeros (for test functions of suitable support) equal the GUE's. While the classical compact groups have identical n-level correlations over all the eigenvalues of a typical matrix, they have distinguishable n-level densities for their eigenvalues near 1.

To any geometric family, the philosophy of Katz and Sarnak ([KS1], [KS2]) states the n-level density depends only on a symmetry group attached to the family. For typical elliptic curve families they predict orthogonal symmetries. One can further analyze the distributions depending on the signs of the functional equations. As our families of elliptic curves are self-dual, we expect the densities to be controlled by the distribution of signs (all even: $SO(\text{even})$; all odd: $SO(\text{odd})$; equidistributed: O).

Previous 1-level density investigations of elliptic curve families were for test functions supported in $(-1, 1)$, where the orthogonal groups' densities are identical. The orthogonal groups have distinguishable (from each other and the other classical compact groups) 2-level densities for functions of arbitrarily small support.

Consider a rational elliptic surface of rank r over $\mathbb{Q}(t)$. Assume GRH (and ABC or the Square-Free Sieve conjecture if $\Delta(t)$ has an irreducible factor of degree ≥ 4). The Birch and Swinnerton-Dyer conjecture and Silverman's Specialization Theorem imply for t large, each curve has r zeros at the critical point. We prove removing these zeros' contributions yield modified densities depending only on the distribution of signs. For all even, odd, and equidistributed, we obtain $SO(\text{even})$, $SO(\text{odd})$ and O as predicted. We verify this for several families of known constant sign.

Let $M(t)$ be the product of the irreducible polynomials dividing $\Delta(t)$ but not $c_4(t)$. Helfgott [He] has shown, assuming standard conjectures, $j(t)$ and $M(t)$ non-constant imply the signs are equidistributed. Thus, for rational elliptic surfaces, the 2-level density provides conditional evidence that the underlying group (in general) is O and not $SO(\text{even})$ or $SO(\text{odd})$. Nevertheless, for small support, we unconditionally verify Katz and Sarnak's conjecture for the 1-level density of a rational elliptic surface and the 2-level density for some families of constant sign.

Finally, we use the 2-level density to obtain better upper bounds on the percent of curves in a family of rank r with rank $r + 2$ or higher, and we explore potential lower order corrections to the densities for several families.

Bibliography

[He] H. Helfgott, *Average root numbers in families of elliptic curves and the average of the Moebius function on integers represented by a polynomial*, preprint.

[ILS] H. Iwaniec, W. Luo and P. Sarnak, *Low lying zeros of families of L-functions*, Inst. Hautes Études Sci. Publ. Math. **91**, 2000, 55 – 131.

[KS1] N. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy*, AMS Colloquium Publications **45**, AMS, Providence, 1999.

[KS2] N. Katz and P. Sarnak, *Zeros of zeta functions and symmetries*, Bull. AMS **36**, 1999, 1 – 26.

[Ru] M. Rubinstein, *Evidence for a spectral interpretation of the zeros of L-functions*, P.H.D. Thesis, Princeton University, 1998, http://www.ma.utexas.edu/users/miker/thesis/thesis.html.

[RS] Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Journal of Math. **81**, 1996, 269 – 322.