Second Discussion Session: July 13^{th}, 2005

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Abstract

Second Discussion Section

All errors should be attributed solely to the typist, Steven J. Miller

1 First

Related to what I'm doing for my thesis, would like to know where to go from here. We have K is a totally real quadratic field, A an ideal class in $\mathcal{O}_K$, $E_{K,A}$ Hilbert modular Eisenstein series. Studying

$$R_A(s_1, s_2, s_3) \;=\; \mathrm{RN}\left(\int_{\mathrm{SL}_2(\mathbb{Z})/\mathcal{H}} E(z, s_0)E(z, s_1)E_{K,A}(z, s_2)\frac{dz}{y^2}\right). \qquad (1)$$

expect some functional equations under s_0 and s_1, can interchange these roles. There are additional, non-obvious functional equations (48). More functional equations then expected.

CAn write as a multiple Dirichlet series by unfolding. Ignoring convergence, three different Eisenstein series to unfold, three different shapes of Dirichlet series to look at. That might be a good place to start.

Take the Rankin convolution of one thing with another - any subset of two and formally write down the product of the sum of the coefficients and see.

Similar to something Bump-Goldfeld: they were interested in the Kronecker formula for totally real fields.

2 Second: Nikos

Want to propose the possibility of another way of producing Multiple Dirichlet Series; will pose as a question. We have the notion of higher order automorphic forms. Define characteristics, many definitions arising in various contexts depending on the applications one has in mind. Define the slash operator $|_k$ acting on some group, and we wan to extend to $\mathbb{C}[\Gamma]$ by linearity. With this convention the defining functional equation for the more generalized automorphic forms is like

$$f|_k(\gamma_1 - 1)(\gamma_2 - 1)\cdots(\gamma_m - 1) \;=\; 0. \qquad (2)$$

We study f satisfying this equation. These are m^{th} order automorphic forms. We expand this product, using the generalization of the slash operator. The motivation for studying these things come from percolation theory, Eisenstein series with modular symbols (introduced by Goldfeld, $E^*(z, s)$), $\mathrm{GL}(2)$ converse theorems, and so on.

The thing which often connects to Multiple Dirichlet Series is that myself and Sreekanan have shown that these higher order automorphic forms are antiderivatives (in a very relative sense) of Chen's iterated integrals (in the 1970s for differential geometric reasons, but it seems to be a natural setup).

More recently Manin-Sreekantan were interested in problems (from iterated integrals) related to multiple L-values, shuffle relations, and polylogarithms. Viewing this in regards to non-commutativity. Manin used these iterated integrals to give some functional equation – he built from these iterated integrals some multiple L-function (though he called it a Multiple Dirichlet Series; originally called it an iterated Mellin transform): $f_1, f_2 \in S_k$ (cusp forms of weight k), and consider

$$L(s_1, s_2, \dots) \;=\; \int_0^\infty f_1(z_1) z_1^{s_1 - 1} \left(\int_\infty^{z_1} f_2(z_2) z_2^{s_2 - 1} \left(\int_\infty^{z_2} \cdots \right) \right) dz_1. \tag{3}$$

Study

$$L(s_1) + L(s_1, s_2) + L(s_1, s_2, s_3). \tag{4}$$

He proved a functional equation for this. The problem that I want to ask about is whether or not we could in some analogous way use the techniques of iterated integrals. Can we make an association from higher order forms to “Perfect” Multiple Dirichlet Series in a way which parallels the classical correspondence between modular forms and L-functions (Dirichlet series, functional equations). The evidence for this is Manin's functional equation (above), the closed relation between iterated integrals and the comments above, and so on. There is one functional equation here – are there enough?

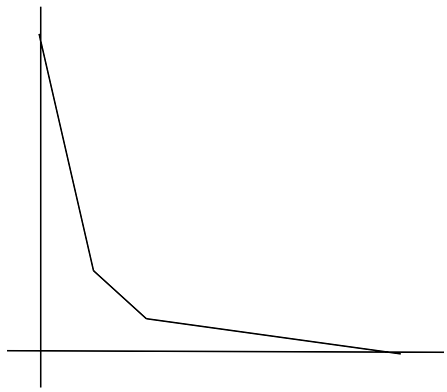
We can construct these directly with the information we have so far. We can classify (myself, Chinta and another) have classified these higher order forms in a fashion. For second order forms we have

$$f_1(z) \int_\infty^z f_2(w) dw. \tag{5}$$

Get in a unique fashion. There are some functional equations, but not enough to produce what we want, the “Perfect” (meaning they can be meromorphically continued) Multiple Dirichlet Series.

3 Third

The question I originally wanted to solve a long time ago was a relationship between certain points that are vertices of a polygon and the genuine non-zero asymptotics associated to Multiple Dirichlet Series.



$$D(s_1, s_2) \;=\; \sum_{\underline{m}\in\mathbb{N}^n} \frac{\phi(\underline{m})}{P_1(\underline{m})^{s_1} P_2(\underline{m})^{s_2}}. \tag{6}$$

We want to show these vertices genuinely determine the asymptotics of the coefficients of the series.

$$N(\pm) \;=\; \sum_{\substack{P_1(\underline{m})=c_1\\ P_2(\underline{m})=c_2}} \phi(\underline{m}). \tag{7}$$

Can only do in two cases. One in which both polynomials are functions of two variables, and the other when very specific.

$$n(\pm) \;=\; \sum_{\substack{P_1(\underline{m})=t_1\\ P_2(\underline{m})=t_2}} \phi(\underline{m}). \tag{8}$$

Discrete problems are harder, so it makes sense to look at it continuously and define

$$E(\underline{t}) \;=\; \int_{P=\underline{t}} \phi \left| \frac{dx_1 \cdots dx_n}{dP_1 \wedge dP_2} \right|. \tag{9}$$

We say $\underline{t_0}$ is a critical value - this means the set is no longer smooth globally, some singular points in the set. What happens to the function as $\underline{t}$ converges to $\underline{t_0}$? If we understand this in generality then can get information

Well known theorem: instead of only worrying about applying ideas in one variable, what about developing a theory of multivariable asymptotic expansions. Hard (hasn't been treated in the literature).

Not talking about singularities of differential equations in several variables. Made progress when just have two functions. Poisson summation over adeles (extending work of others) – are there any interesting applications.

Theorem 3.1. *Let $P = (P_1, P_2)$ where P_i is homogeneous of degree $d_i \geq 2$ (over $\mathbb{Q}$) with the property that each P_i has isolated singular point in $\overline{\mathbb{Q}}^n$. Let $\mathcal{C}_P$ be the critical locus of P; it is the union of smooth (i.e., reduced) lines. It is a homogeneous set so we have lines over $\overline{\mathbb{Q}}$. We have the critical point where things are together, and a continuous set of points where we lose regularity. It seems to be a fairly tricky thing to deal with analytically.*

Assume additionally that $n > N_P$, where n is the number of variables and

$$N_P = \begin{cases} \max\{3d_1, 3d_2\} & \text{if there is an } i \text{ with } d_i \geq 3 \\ 8 & \text{if } d_1 = d_2 = 2. \end{cases} \tag{10}$$

Then in this case we have a Poisson summation formula over $\mathbb{A}_{\mathbb{Q}}^2$ of the following form:

$$\sum_{\underline{t} \in \iota\mathbb{Q}^2} F(\underline{t}, \phi) = \sum_{\underline{\lambda} \in \iota\mathbb{Q}^2} F^*(\lambda, \phi), \tag{11}$$

where ϕ is in the Schwartz space of $X_{\mathbb{A}_{\mathbb{Q}}}$.

This is a result of five articles - must understand the singular behavior, Fourier analysis in two variables uniformly, ...

Question: for one quadratic polynomial in $n \geq 5$ variables, this type of formula plus metaplectic theory allows us to show that the Hasse principle applies to the hypersurface $P = 0$. That is a non-singular solution in each $\mathbb{Q}_p$ implies the existence of a non-trivial integer solution. Does this type of formula and an extension of the theory of Weil imply that we can prove a Hasse principle for the intersection of two surfaces $P_1 = 0$ and $P_2 = 0$?

4 Fourth

Want to introduce a new class of global Dirichlet series attached to function fields. These are really generalizations of the Multiple Dirichlet Series defined by Zagier in the early 1990s. He defines

$$\zeta_d(s_1, \ldots, s_d) = \sum_{0<n_1<n_2<\cdots<n_d} n_1^{-s_1} \cdots n_d^{-s_d}. \tag{12}$$

Can view as a pairing of a distribution and a test function, analytic continuation of certain functions. Interest in this function stems largely among the multiple zeta values. Region of absolute convergence: fix a_k to e an integer greater than or equal to 1, with $a_d > 1$. Study $\zeta_d(a_1, \ldots, a_d)$. See the works of Goncharov, Lontsevich, Manin, Zagier. Will look at function field analogue and see how that gives rise to rational functions.

The simplest possible setting is the ring of polynomials over a finite field. Let $\mathbb{F}_q$ be the finite field with q elements, consider $\mathbb{F}_q[T]$ and set $|f| = q^{\deg(f)}$. We study

$$Z(s) = \sum_{\substack{F \in \mathbb{F}_q[T] \\ \text{monic}}} |f|^{-s}, \tag{13}$$

which converges for $\mathfrak{Re}(s) > 1$ to

$$Z(s) = \frac{1}{1 - q^{1-s}}. \tag{14}$$

Let

$$Z(s_1, \ldots, s_d) = \sum_{\substack{(f_1, \ldots, f_d) \in \mathbb{F}_q[T]^d \\ 0<\deg(f_1) \leq \cdots \leq \deg(f_d)}} \prod_{k=1}^{d} |f_k|^{-s_k}. \tag{15}$$

The first result is

Theorem 4.1. *For the polynomial ring we have*

$$Z(s_1, \ldots, s_d) = \prod_{k=1}^{d} Z(s_1 + \cdots + s_d - (d-k)). \tag{16}$$

This implies a meromorphic continuation, rational function in $q^{-s_1}, \ldots, q^{-s_d}$*, and has an Euler product. Can conclude it has a functional equation in the first variable.*

We have (after multiplying factors together)

$$\prod_{k=1}^{d} \frac{q^{-(s_1+\cdots+s_d)-(d-k)}}{1-q^{-(s_1+\cdots+s_d)}} Z(s_1,\ldots,s_d) \tag{17}$$

Let K be a global function field of genus g, D a divisor of K, and $\mathcal{D}_K^+$ the effective divisors. The norm of a divisor is $|D| = q^{\deg(D)}$.

For the function field, the zeta function is defined by a sum over effective divisor:

$$Z(s_1,\ldots,s_d) \;=\; \sum_{\substack{(D_1,\ldots,D_d)\\ 0<\det(D_1)\le\cdots\le\deg(D_d)}} \prod_{k=1}^{d} |D_k|^{-s_k}. \tag{18}$$

Suppose the genus $g=0$. Then there is a polynomial such that $P(q^{-s_1},\ldots,q^{-s_d})Z(s_1,\ldots,s_d)$ is a polynomial of degree at most $2d-1$ in the q^{s_j}, where

$$P(q^{-s_1},\ldots,q^{-s_d}) \;=\; (q-1)^d(1-q^{-sd})(1-q^{1-sd})\prod_{k=1}^{d}(1-q^{s_k+\cdots+s_d})(1-q^{1-()})(1-q^{2-()}). \tag{19}$$

If we set $K=\mathbb{F}_2[T]$ and $d=2$ then we get products of simple zeta functions

$$\begin{aligned} Z(s,w) \quad = \quad & \frac{q^2}{(q-1)^2}Z(s+w-1)Z(w) - \frac{q}{(q-1)^2}Z(s+w)Z(w+1) \\ & -\frac{q}{(q-1)^2}Z(s+w)Z(w) + \frac{1}{(q-1)^2}Z(s+w+1)Z(w+1). \end{aligned} \tag{20}$$

Similar things for arbitrary genus.

5 Fifth

6 Sixth