Powers of Algebraic Disjointness Preserving Operators

Sam Watson
Gerard Buskes, Advisor

University of Mississippi
Department of Mathematics
Master's Thesis Defense

April 30, 2009
Disjoint vectors

These two vectors in $\mathbb{R}^{10}$ have the property that in every position, at least one of them has a zero entry.

\[
\begin{pmatrix}
0 \\
1 \\
0 \\
9 \\
4 \\
0 \\
5 \\
0 \\
0 \\
1
\end{pmatrix}
\quad
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
3 \\
0 \\
4 \\
2 \\
0
\end{pmatrix}
\]
These two vectors in $\mathbb{R}^{10}$ have the property that in every position, at least one of them has a zero entry.

$$
\begin{pmatrix}
0 \\
1 \\
0 \\
9 \\
4 \\
0 \\
5 \\
0 \\
0 \\
1
\end{pmatrix} \quad \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0 \\
3 \\
0 \\
4 \\
2 \\
0
\end{pmatrix}
$$

This property is called \textit{disjointness}, and if $u$ and $v$ are disjoint, we write $u \perp v$. 
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$
T = 
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix \textit{disjointness preserving}, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

\[
T^2 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$
T^3 = 
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

\[
T^5 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^6 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^7 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix \textit{disjointness preserving}, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^8 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$
T^9 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^{10} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix \textit{disjointness preserving}, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^{11} = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^{12} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perpTv$. Observe what happens when we compute powers of $T$.

$$T^2 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$
The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^3 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$
The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^5 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^6 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \implies Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^7 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

\[
T^8 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^9 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$T^{10} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

$$
T^{11} = 
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$
A Disjointness Preserving Matrix

The following matrix $T$ has a most one nonzero entry in each row. We call such a matrix *disjointness preserving*, since $u \perp v \Rightarrow Tu \perp Tv$. Observe what happens when we compute powers of $T$.

\[
T^{12} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Let us visualize the movement of the nonzero entries in the previous matrix using a directed graph. For $1 \leq i \leq 10$ and $1 \leq j \leq 10$, we connect $i$ to $j$ if the $(i,j)$th entry of the matrix is nonzero.
Let us visualize the movement of the nonzero entries in the previous matrix using a directed graph. For $1 \leq i \leq 10$ and $1 \leq j \leq 10$, we connect $i$ to $j$ if the $(i, j)$th entry of the matrix is nonzero.

![Diagram of directed graph]
Let us visualize the movement of the nonzero entries in the previous matrix using a directed graph. For $1 \leq i \leq 10$ and $1 \leq j \leq 10$, we connect $i$ to $j$ if the $(i,j)$th entry of the matrix is nonzero.

$T^{12}$ is diagonal on the matrix obtained by dropping the second, third, and fifth rows and columns because 12 is a common multiple of the cycle lengths of the graph.
Let us visualize the movement of the nonzero entries in the previous matrix using a directed graph. For $1 \leq i \leq 10$ and $1 \leq j \leq 10$, we connect $i$ to $j$ if the $(i,j)$th entry of the matrix is nonzero.

$T^{12}$ is diagonal on the matrix obtained by dropping the second, third, and fifth rows and columns because 12 is a common multiple of the cycle lengths of the graph. Also, 2, 3, and 5 are not in a cycle.
The graph gives a complete description of $T$ as an operator from $\mathbb{R}^d$ to $\mathbb{R}^d$. 

...
The graph gives a complete description of $T$ as an operator from $\mathbb{R}^d$ to $\mathbb{R}^d$.

Let $\tau(k)$ be the vertex to which $k$ is connected. The $k$th entry of $Tv$ is $\tau(k)$th entry of $v$. 
Digraph Representation

- The graph gives a complete description of $T$ as an operator from $\mathbb{R}^d$ to $\mathbb{R}^d$.
- Let $\tau(k)$ be the vertex to which $k$ is connected. The $k$th entry of $Tv$ is $\tau(k)$th entry of $v$.
- Similarly, the $k$th entry of $T^rv$ is $v_{\tau^r(k)}$. 

If $T$ has entries other than 0 and 1 (but still has at most one nonzero entry in each row), then a weighted digraph describes the action of $T$. Simply associate with each edge $(k, \tau(k))$ a weight: the $(k, \tau(k))$th entry of $T$. The $k$th entry of $T^rv$ is $v_{\tau^r(k)}$ times the product of the weights in the path from $k$ to $\tau^r(k)$. 

Sam Watson

Powers of Algebraic Disjointness Preserving Operators
Digraph Representation

- The graph gives a complete description of $T$ as an operator from $\mathbb{R}^d$ to $\mathbb{R}^d$.
- Let $\tau(k)$ be the vertex to which $k$ is connected. The $k$th entry of $Tv$ is $\tau(k)$th entry of $v$.
- Similarly, the $k$th entry of $T^r v$ is $v_{\tau^r(k)}$.
- If $T$ has entries other than 0 and 1 (but still has at most one nonzero entry in each row), then a weighted digraph describes the action of $T$. 
The graph gives a complete description of $T$ as an operator from $\mathbb{R}^d$ to $\mathbb{R}^d$.

Let $\tau(k)$ be the vertex to which $k$ is connected. The $k$th entry of $Tv$ is $\tau(k)$th entry of $v$.

Similarly, the $k$th entry of $T^r v$ is $v_{\tau^r(k)}$.

If $T$ has entries other than 0 and 1 (but still has at most one nonzero entry in each row), then a weighted digraph describes the action of $T$.

Simply associate with each edge $(k, \tau(k))$ a weight: the $(k, \tau(k))$th entry of $T$. 
Digraph Representation

- The graph gives a complete description of $T$ as an operator from $\mathbb{R}^d$ to $\mathbb{R}^d$.
- Let $\tau(k)$ be the vertex to which $k$ is connected. The $k$th entry of $Tv$ is $\tau(k)$th entry of $v$.
- Similarly, the $k$th entry of $T^r v$ is $v_{\tau^r(k)}$.
- If $T$ has entries other than 0 and 1 (but still has at most one nonzero entry in each row), then a weighted digraph describes the action of $T$.
- Simply associate with each edge $(k, \tau(k))$ a weight: the $(k, \tau(k))$th entry of $T$.
- The $k$th entry of $T^r v$ is $v_{\tau^r(k)}$ times the product of the weights in the path from $k$ to $\tau^r(k)$. 
Digraph Representation: An Example

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
3 \\
6 \\
4 \\
0 \\
1 \\
5 \\
2 \\
5 \\
6 \\
0 \\
5
\end{pmatrix}
= 
\begin{pmatrix}
6 \\
1 \\
6 \\
3 \\
6 \\
2 \\
5 \\
0 \\
0 \\
5
\end{pmatrix}
\]
The Theorem Statement for Matrices

**Theorem 1**

Let $T \in \mathbb{R}^{d \times d}$ be a disjointness preserving matrix, let $G$ be the digraph associated with $T$. Let $C := \{1 \leq k \leq n : k \text{ is in a cycle in } G\}$. If $M$ is divisible by all the cycle lengths in $G$, then $T^M$ is the identity matrix on the subspace of $\mathbb{R}^d$ generated by $\{e_k\}_{k \in C}$.

- Instead of proving this theorem directly, we look to generalize.
The Theorem Statement for Matrices

**Theorem 1**

Let \( T \in \mathbb{R}^{d \times d} \) be a disjointness preserving matrix, let \( G \) be the digraph associated with \( T \). Let \( C := \{1 \leq k \leq n : k \) is in a cycle in \( G \}\). If \( M \) is divisible by all the cycle lengths in \( G \), then \( T^M \) is the identity matrix on the subspace of \( \mathbb{R}^d \) generated by \( \{e_k\}_{k \in C} \).

- Instead of proving this theorem directly, we look to generalize.
- Since \( T \) represents a linear operator from \( \mathbb{R}^d \) to \( \mathbb{R}^d \), we first consider an arbitrary linear operator \( T : L \rightarrow L \), where \( L \) is any vector space.
Theorem 1

Let $T \in \mathbb{R}^{d \times d}$ be a disjointness preserving matrix, let $G$ be the digraph associated with $T$. Let $C := \{1 \leq k \leq n : k \text{ is in a cycle in } G\}$. If $M$ is divisible by all the cycle lengths in $G$, then $T^M$ is the identity matrix on the subspace of $\mathbb{R}^d$ generated by $\{e_k\}_{k \in C}$.

Instead of proving this theorem directly, we look to generalize.

Since $T$ represents a linear operator from $\mathbb{R}^d$ to $\mathbb{R}^d$, we first consider an arbitrary linear operator $T : L \rightarrow L$, where $L$ is any vector space.

We need to abstract the condition of disjointness.
Vector Lattices

- We define a partial order on $\mathbb{R}^d$ coordinate-wise. We say $u \leq v$ in $\mathbb{R}^d$ if and only if $u_k \leq v_k$ for $k = 1, 2, \ldots, d$. 

...
We define a partial order on $\mathbb{R}^d$ coordinate-wise. We say $u \leq v$ in $\mathbb{R}^d$ if and only if $u_k \leq v_k$ for $k = 1, 2, \ldots, d$.

This partial order respects addition and scalar multiplication.
We define a partial order on $\mathbb{R}^d$ coordinate-wise. We say $u \leq v$ in $\mathbb{R}^d$ if and only if $u_k \leq v_k$ for $k = 1, 2, \ldots, d$.

This partial order respects addition and scalar multiplication:

- $u \leq v \Rightarrow u + w \leq v + w$ for all $w \in \mathbb{R}^d$, and
Vector Lattices

- We define a partial order on $\mathbb{R}^d$ coordinate-wise. We say $u \leq v$ in $\mathbb{R}^d$ if and only if $u_k \leq v_k$ for $k = 1, 2, \ldots, d$.
- This partial order respects addition and scalar multiplication:
  - $u \leq v \Rightarrow u + w \leq v + w$ for all $w \in \mathbb{R}^d$, and
  - $0 \leq v$ and $0 \leq \lambda \Rightarrow 0 \leq \lambda v$. 
Vector Lattices

- We define a partial order on $\mathbb{R}^d$ coordinate-wise. We say $u \leq v$ in $\mathbb{R}^d$ if and only if $u_k \leq v_k$ for $k = 1, 2, \ldots, d$.
- This partial order respects addition and scalar multiplication:
  - $u \leq v \Rightarrow u + w \leq v + w$ for all $w \in \mathbb{R}^d$, and
  - $0 \leq v$ and $0 \leq \lambda \Rightarrow 0 \leq \lambda v$.
- In addition, $\mathbb{R}^d$ with coordinate-wise order is a lattice, i.e. every pair of elements $v$ and $w$ has a least upper bound (denoted $v \vee w$) and a greatest lower bound (denoted $v \wedge w$).
We define a partial order on $\mathbb{R}^d$ coordinate-wise. We say $u \leq v$ in $\mathbb{R}^d$ if and only if $u_k \leq v_k$ for $k = 1, 2, \ldots, d$.

This partial order respects addition and scalar multiplication:
- $u \leq v \Rightarrow u + w \leq v + w$ for all $w \in \mathbb{R}^d$, and
- $0 \leq v$ and $0 \leq \lambda \Rightarrow 0 \leq \lambda v$.

In addition, $\mathbb{R}^d$ with coordinate-wise order is a lattice, i.e. every pair of elements $v$ and $w$ has a least upper bound (denoted $v \lor w$) and a greatest lower bound (denoted $v \land w$).

A partially ordered vector space $(L, \leq)$ whose order structure is a lattice is called a vector lattice, or Riesz space.
Example

For any topological space $X$, the set $C(X)$ of continuous functions from $X$ to $\mathbb{R}$ is a Riesz space under pointwise addition, scalar multiplication and order.

- If $X$ is equipped with the discrete topology, then $C(X)$ is denoted $\mathbb{R}^X$. 
Vector Lattices

Example

For any topological space $X$, the set $C(X)$ of continuous functions from $X$ to $\mathbb{R}$ is a Riesz space under pointwise addition, scalar multiplication and order.

- If $X$ is equipped with the discrete topology, then $C(X)$ is denoted $\mathbb{R}^X$.
- In particular, $\mathbb{R}^d$ is $C(\{1, 2, 3 \ldots, d\})$. 
Example

For any topological space $X$, the set $C(X)$ of continuous functions from $X$ to $\mathbb{R}$ is a Riesz space under pointwise addition, scalar multiplication and order.

- If $X$ is equipped with the discrete topology, then $C(X)$ is denoted $\mathbb{R}^X$.
- In particular, $\mathbb{R}^d$ is $C(\{1, 2, 3 \ldots, d\})$.
- In $C(X)$, $0 \leq nu \leq v$ for all $n \in \mathbb{N} \Rightarrow u = 0$. Any Riesz space satisfying this condition is called Archimedean.
In $C(X)$, $u \perp v$ if and only if $|u| \wedge |v| = 0$. 
In $C(X)$, $u \perp v$ if and only if $|u| \wedge |v| = 0$.
We take $|u| \wedge |v| = 0$ as the definition of disjointness in an Riesz space $L$. 
In $\mathcal{C}(X)$, $u \perp v$ if and only if $|u| \wedge |v| = 0$.

We take $|u| \wedge |v| = 0$ as the definition of disjointness in an Riesz space $L$.

Can we prove a generalization of Theorem 1 for disjointness preserving operators on vector lattices?
In $C(X)$, $u \perp v$ if and only if $|u| \wedge |v| = 0$.

We take $|u| \wedge |v| = 0$ as the definition of disjointness in an Riesz space $L$.

Can we prove a generalization of Theorem 1 for disjointness preserving operators on vector lattices?

Answer:
In $C(X)$, $u \perp v$ if and only if $|u| \wedge |v| = 0$.

We take $|u| \wedge |v| = 0$ as the definition of disjointness in an Riesz space $L$.

Can we prove a generalization of Theorem 1 for disjointness preserving operators on vector lattices?

Answer: no
In $C(X)$, $u \perp \nu$ if and only if $|u| \wedge |\nu| = 0$.

We take $|u| \wedge |\nu| = 0$ as the definition of disjointness in an Riesz space $L$.

Can we prove a generalization of Theorem 1 for disjointness preserving operators on vector lattices?

Answer: no

Setting up the graph requires more conditions on the operator $T$. 
Algebraic Operators

The Cayley-Hamilton theorem in linear algebra implies that every (finite-dimensional) matrix satisfies a polynomial equation, in other words there exist real numbers $a_0, a_1, \ldots, a_n$ for which

$$T^n + a_{n-1}T^{n-1} + \cdots + a_1T + a_0$$

is the zero matrix.
The Cayley-Hamilton theorem in linear algebra implies that every (finite-dimensional) matrix satisfies a polynomial equation, in other words there exist real numbers $a_0, a_1, \ldots, a_n$ for which

$$T^n + a_{n-1}T^{n-1} + \cdots + a_1T + a_0$$

is the zero matrix.

If $L$ is a vector space, an operator $T : L \rightarrow L$ is said to be algebraic if it satisfies a polynomial equation.
The Cayley-Hamilton theorem in linear algebra implies that every (finite-dimensional) matrix satisfies a polynomial equation, in other words there exist real numbers $a_0, a_1, \ldots, a_n$ for which

$$T^n + a_{n-1} T^{n-1} + \cdots + a_1 T + a_0$$

is the zero matrix.

If $L$ is a vector space, an operator $T : L \rightarrow L$ is said to be **algebraic** if it satisfies a polynomial equation.

For any algebraic operator $T : L \rightarrow L$, there is a unique monic polynomial $p$ of minimum degree for which $p(T)$ is the zero operator.
Algebraic Operators

- The Cayley-Hamilton theorem in linear algebra implies that every (finite-dimensional) matrix satisfies a polynomial equation, in other words there exist real numbers $a_0, a_1, \ldots, a_n$ for which

$$T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0$$

is the zero matrix.

- If $L$ is a vector space, an operator $T : L \to L$ is said to be algebraic if it satisfies a polynomial equation.

- For any algebraic operator $T : L \to L$, there is a unique monic polynomial $p$ of minimum degree for which $p(T)$ is the zero operator.

- This polynomial is called the minimal polynomial of $T$. 
Example

Let $L = \mathbb{R}^\infty$, the set of sequences of real numbers with pointwise order. Define $T(a_1, a_2, a_3 \ldots) := (0, a_1, a_2, \ldots)$. $T$ is not an algebraic operator.
Algebraic Operators

Example

Let $L = \mathbb{R}^\mathbb{N}$, the set of sequences of real numbers with pointwise order. Define $T(a_1, a_2, a_3 \ldots) := (0, a_1, a_2, \ldots)$. $T$ is not an algebraic operator.

To see that $T$ is not an algebraic, let

$$p(T) = T^n + a_{n-1} T^{n-1} + \cdots + a_1 T + a_0$$

be a polynomial. Let $m$ be the least index for which $a_m \neq 0$, and let $u = (1, 0, 0, 0, \ldots)$. For all $m < r \leq n$, we have $(a_r T^r u)_m = 0$. Since $(a_m T^m u)_m \neq 0$, the vector $p(T)u$ is not zero in the $m$th position and hence $p(T)$ cannot be the zero operator.
A subset $A$ of a Riesz space $L$ is said to be order bounded if there exist $u, v \in L$ with $u \leq x \leq v$ for all $x \in A$. 
Order Bounded Operators

- A subset $A$ of a Riesz space $L$ is said to be *order bounded* if there exist $u, v \in L$ with $u \leq x \leq v$ for all $x \in A$.
- An operator $T$ from $L$ to another Riesz space $M$ is called an *order bounded operator* if it maps order bounded sets to order bounded sets.
A subset $A$ of a Riesz space $L$ is said to be *order bounded* if there exist $u, v \in L$ with $u \leq x \leq v$ for all $x \in A$.

An operator $T$ from $L$ to another Riesz space $M$ is called an *order bounded operator* if it maps order bounded sets to order bounded sets.

In other words, $T$ is order bounded if $T[A]$ is order bounded in $M$ for every order bounded $A \subset L$. 

Example: If $(X, \mu)$ is a measure space, then the operator from $L^1(\mu)$ to $\mathbb{R}$ defined by $f \mapsto \int f \, d\mu$ is order bounded, since a function $g$ which is bounded above by $h \in L^1(\mu)$ and below by $f \in L^1(\mu)$ has $\int f \, d\mu \leq \int g \, d\mu \leq \int h \, d\mu$. 

Sam Watson

Powers of Algebraic Disjointness Preserving Operators
Order Bounded Operators

- A subset $A$ of a Riesz space $L$ is said to be order bounded if there exist $u, v \in L$ with $u \leq x \leq v$ for all $x \in A$.
- An operator $T$ from $L$ to another Riesz space $M$ is called an order bounded operator if it maps order bounded sets to order bounded sets.
- In other words, $T$ is order bounded if $T[A]$ is order bounded in $M$ for every order bounded $A \subset L$.

**Example**

If $(X, \mu)$ is a measure space, then the operator from $L^1(\mu)$ to $\mathbb{R}$ defined by $f \mapsto \int f \, d\mu$ is order bounded, since a function $g$ which is bounded above by $h \in L^1(\mu)$ and below by $f \in L^1(\mu)$ has $\int f \, d\mu \leq \int g \, d\mu \leq \int h \, d\mu$. 
Order Bounded Operators

Example

A function $f : [0, \infty) \to \mathbb{R}$ is *essentially polynomial* if there is a positive real number $x_f$ and a polynomial $p_f$ so that for all $x > x_f$, $p_f(x) = f(x)$. The set $L$ of all essentially polynomial functions on $[0, \infty)$ is an Riesz space (under pointwise order). The function $f \mapsto p_f(0)$ is a disjointness-preserving operator on $L$ which is not order bounded.
Example

A function $f : [0, \infty) \to \mathbb{R}$ is *essentially polynomial* if there is a positive real number $x_f$ and a polynomial $p_f$ so that for all $x > x_f$, $p_f(x) = f(x)$. The set $L$ of all essentially polynomial functions on $[0, \infty)$ is an Riesz space (under pointwise order). The function $f \mapsto p_f(0)$ is a disjointness-preserving operator on $L$ which is not order bounded.

To see that $f \mapsto p_f(0)$ is not order bounded, define

$$f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq n \\ -4n(x - n) & \text{if } n < x. \end{cases}$$
**Example**

A function $f : [0, \infty) \to \mathbb{R}$ is *essentially polynomial* if there is a positive real number $x_f$ and a polynomial $p_f$ so that for all $x > x_f$, $p_f(x) = f(x)$. The set $L$ of all essentially polynomial functions on $[0, \infty)$ is an Riesz space (under pointwise order). The function $f \mapsto p_f(0)$ is a disjointness-preserving operator on $L$ which is not order bounded.

To see that $f \mapsto p_f(0)$ is not order bounded, define

$$f_n(x) = \begin{cases} 
0 & \text{if } 0 \leq x \leq n \\
-4n(x - n) & \text{if } n < x.
\end{cases}$$

The set $\{f_n\}_{n \in \mathbb{N}}$ is order bounded, since $-x^2 \leq f_n \leq 0$. However, $p_{f_n}(0) = 4n^2$, so the set $\{p_{f_n}(0)\}_{n \in \mathbb{N}}$ of images of $\{f_n\}_{n \in \mathbb{N}}$ under $f \mapsto p_f(0)$ is not an order bounded subset of $\mathbb{R}$. 
Orthomorphisms

- We introduce the property of operators on Riesz spaces which generalizes the concept of a diagonal operator on $\mathbb{R}^d$. 

We introduce the property of operators on Riesz spaces which generalizes the concept of a diagonal operator on $\mathbb{R}^d$.

If $T : L \to L$ is an order bounded operator for which $u \perp v \implies Tu \perp v$ for all $u, v \in L$, then $T$ is called an orthomorphism.
We introduce the property of operators on Riesz spaces which generalizes the concept of a diagonal operator on $\mathbb{R}^d$.

If $T : L \to L$ is an order bounded operator for which $u \perp v \implies Tu \perp v$ for all $u, v \in L$, then $T$ is called an orthomorphism.

If $A$ is a subspace of $T$ for which $Tu \in A$ for all $u \in A$, then $A$ is said to be $T$-invariant. $T|_A$ may be viewed as an operator from $A$ to $A$. 
Orthomorphisms

- We introduce the property of operators on Riesz spaces which generalizes the concept of a diagonal operator on $\mathbb{R}^d$.

- If $T : L \to L$ is an order bounded operator for which $u \perp v \implies Tu \perp v$ for all $u, v \in L$, then $T$ is called an orthomorphism.

- If $A$ is a subspace of $T$ for which $Tu \in A$ for all $u \in A$, then $A$ is said to be $T$-invariant. $T|_A$ may be viewed as an operator from $A$ to $A$.

- Let $L$ be a Riesz space, let $T$ be an order bounded operator $T : L \to L$ and let $A$ be a $T$-invariant space of $L$. If $u \perp v \implies Tu \perp v$ for all $u, v \in A$, then $T$ is called an orthomorphism on $A$. 
We use the following theorem to set up the digraph associated with $T$. We restrict our attention to \textit{realcompact} topological spaces, which includes “most” topological spaces typically encountered.
Setting up the graph

We use the following theorem to set up the digraph associated with $T$. We restrict our attention to *realcompact* topological spaces, which includes “most” topological spaces typically encountered.

**Theorem 2**

If $X$ is a realcompact topological space and $T$ is an order bounded disjointness preserving operator on $C(X)$, then for all $x \in X$, there either $\delta_x \circ T$ is the zero functional or there exists a unique $y \in X$ so that

$$\delta_x \circ T = ((\delta_x \circ T)(1)) \delta_y.$$
Setting up the graph

We use the following theorem to set up the digraph associated with $T$. We restrict our attention to \textit{realcompact} topological spaces, which includes “most” topological spaces typically encountered.

**Theorem 2**

If $X$ is a realcompact topological space and $T$ is an order bounded disjointness preserving operator on $C(X)$, then for all $x \in X$, there either $\delta_x \circ T$ is the zero functional or there exists a unique $y \in X$ so that

$$
\delta_x \circ T = ((\delta_x \circ T)(1)) \delta_y.
$$

Call the unique $y$ described in the theorem $\tau(x)$ (if $\delta_x \circ T$ is nonzero) and define the digraph associated with $T$ to have vertex set $X$ and weighted edge set

$$
\left\{ \left( (x, \tau(x)), (\delta_x \circ T)(1) \right) : \delta_x \circ T \neq 0 \right\}.
$$
Setting up the graph

We get a representation of powers of $T$ from the graph as we did for operators on $\mathbb{R}^d$. Define $\lambda(x)$ to be the maximum $r$ for which $\tau^r(x)$ is defined, and define the product-of-weights function

$$w_r(x) = \begin{cases} 
\prod_{i=1}^{r}(T1)(\tau^i(x)) & \text{if } r \leq \lambda(x) \\
0 & \text{if } r > \lambda(x) 
\end{cases}$$

**Proposition**

If $T$ is an order bounded disjointness preserving operator on $C(X)$, and $r \geq 0$ is an integer, then for all $x \in X$,

$$(T^rf)(x) = \begin{cases} 
f(\tau^r(x))w_r(x) & \text{if } r \leq \lambda(x) \\
0 & \text{if } r > \lambda(x) \end{cases}.$$
Preliminary Remarks

We generalize Theorem 1 to operators on spaces of the form $C(X)$ before generalizing to Archimedean Riesz spaces. Three remarks:
Preliminary Remarks

We generalize Theorem 1 to operators on spaces of the form $C(X)$ before generalizing to Archimedean Riesz spaces. Three remarks:

- We state the theorem for $X$ realcompact. If $X$ is not realcompact, we may apply the following theorem to the realcompactification $\nu X$ of $X$. 

*Sam Watson*
Preliminary Remarks

We generalize Theorem 1 to operators on spaces of the form $C(X)$ before generalizing to Archimedean Riesz spaces. Three remarks:

- We state the theorem for $X$ realcompact. If $X$ is not realcompact, we may apply the following theorem to the realcompactification $\nu X$ of $X$.

- We will get a larger subspace on which $T^M$ is a diagonal operator, namely the range of $T^m$, where $m$ is the smallest index for which the coefficient of $T^m$ in the minimal polynomial of $T$ is nonzero.
Preliminary Remarks

We generalize Theorem 1 to operators on spaces of the form $C(X)$ before generalizing to Archimedean Riesz spaces. Three remarks:

- We state the theorem for $X$ realcompact. If $X$ is not realcompact, we may apply the following theorem to the realcompactification $\nu X$ of $X$.

- We will get a larger subspace on which $T^M$ is a diagonal operator, namely the range of $T^m$, where $m$ is the smallest index for which the coefficient of $T^m$ in the minimal polynomial of $T$ is nonzero.

- We have to ensure that $M \geq m$ so that all the paths are long enough that $\tau^M(x)$ is in a cycle.
Theorem 3
Let $X$ be a realcompact topological space, and let $T$ be an algebraic order bounded disjointness preserving operator on $C(X)$. Let $m$ denote the multiplicity of zero as a root of the minimal polynomial of $T$. If $M \geq m$ is any positive integer divisible by each of the cycle lengths of the graph associated with $T$, then the restriction of $T^M$ to the range of $T^m$ is an orthomorphism.
Corollaries

Corollary

If $T$ is a bijective algebraic order bounded disjointness preserving operator on $C(\mathbb{R})$ for which $T(1) = 1$, then the minimal polynomial of $T$ is of the form $x^{2p} - 1$ for some nonnegative integer $p$. 
Corollary

Let $X$ be a completely regular extremally disconnected compact Hausdorff space, and let $T$ be an algebraic bijective order bounded disjointness preserving operator on $C(X)$ whose minimal polynomial has degree $n$. Then each set $X_k = c^{-}[\{k\}]$ (for $k = 1, 2, \cdots, n$) is both open and closed, and for each $k$, $B_k = \{f \in C(X) : f = 0 \text{ on } X \setminus X_k\}$ is a band in $C(X)$. Let $K \subset \{1, 2, \ldots, n\}$ consist of the indices for which $X_k$ is nonempty. Then $C(X) = \bigoplus_{k \in K} B_k$. Moreover, $T$ is invariant on each band $B_k$, $T^k$ is an orthomorphism on $B_k$, and for all $f \in C(X)$ we have

$$Tf = \sum_{k \in K} \left( T\big|_{B_k} \right) f_k = \bigvee_{k \in K} \left( T\big|_{B_k} \right) f_k,$$
Theorem 3

If $T$ is an algebraic order bounded disjointness preserving operator on an Archimedean Riesz space $L$ with minimal polynomial $p(T) = T^n + \cdots + a_m T^m$ and $M \geq m$ is a multiple of $1, 2, \ldots, n - m$, then $T^M$ restricted to the range of $T^m$ is an orthomorphism.

The proof uses Kakutani’s representation theorem to find $C(X)$ which represents $L$ locally.
Thank you!