

Math 0100: Introductory Calculus II Practice Final Solutions

It's in your best interest to not look at the solutions until you've taken the practice test under timed, test-taking conditions. Just a heads up.

1) Determine whether

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

is absolutely convergent, conditionally convergent or divergent. Justify your answer.

Answer: We see that $n \ln n$ increases to infinity so $1/(n \ln n)$ decreases to 0, thus the alternating series test tells us that the series is convergent. It remains to determine if the series is absolutely convergent. It is worth noting that the ratio test fails, and since we can't compare this series with a p -series our only remaining strategy is the integral test. This series above is absolutely convergent if and only if

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

is convergent, since $1/(n \ln n)$ is positive and decreasing, the integral test tells us that this series converges if and only if

$$\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x \ln x} dx$$

converges. We use the substitution $u = \ln x$ to get $du = dx/x$ and so

$$\lim_{t \rightarrow \infty} \int_2^t \frac{1}{x \ln x} dx = \lim_{t \rightarrow \infty} \int_{\ln 2}^{\ln t} \frac{du}{u} = \lim_{t \rightarrow \infty} [\ln u]_{\ln 2}^{\ln t} = \lim_{t \rightarrow \infty} (\ln(\ln t) - \ln(\ln 2)) = \infty.$$

So the integral test tells us that $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ is divergent, thus our original series is only conditionally convergent.

2) Express

$$\frac{1}{1 + 27x^3}$$

as a Maclaurin series, and give the interval of convergence.

Answer: We recall the Maclaurin series for the geometric series:

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$$

exactly when $|x| < 1$ and so

$$\frac{1}{1 + 27x^3} = \frac{1}{1 - (-27x^3)} = \sum_{n=0}^{\infty} (-27x^3)^n = \sum_{n=0}^{\infty} (-27)^n x^{3n}$$

when $|-27x^3| = (3|x|)^3 < 1$, and taking the cube-root of both sides we get $3|x| < 1$ so $|x| < 1/3$. Thus we have the Maclaurin series $\sum_{n=0}^{\infty} (-27)^n x^{3n}$ in the interval $(-1/3, 1/3)$.

3) Use either multiplication or division of power series to give the first three nonzero terms in the Maclaurin series for the function

$$\frac{e^x}{1-x}$$

Answer: We can think of this as

$$e^x \left(\frac{1}{1-x} \right) = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} x^n \right) = \left(1 + x + \frac{x^2}{2} + \dots \right) (1 + x + x^2 + \dots)$$

and multiply the first few terms to get

$$1(1+x+x^2+\dots) + x(1+x+x^2+\dots) + \frac{x^2}{2}(1+x+x^2+\dots) + \dots$$

and only worry about the terms up to second degree (since all the coefficients are nonzero) so we have

$$(1+x+x^2+\dots) + (x+x^2+\dots) + \left(\frac{x^2}{2} + \dots \right) = 1 + 2x + \frac{5}{2}x^2 + \dots$$

and we have the desired result.

The other option is long division:

$$\begin{array}{r} 1 + 2x + \frac{5}{2}x^2 + \dots \\ 1-x \overline{) 1 + x + \frac{x^2}{2} + \dots} \\ - (1 - x) \\ \hline 2x + \frac{x^2}{2} + \dots \\ - (2x - 2x^2) \\ \hline \frac{5x^2}{2} + \dots \end{array}$$

which gives the same result.

4) Find the centroid of the region bounded by $y = 2x$, $y = x/2$ and $x = 2$.

Answer: Drawing a picture is probably helpful. First we need to compute the area of the region

$$A = \int_0^2 \left(2x - \frac{x}{2} \right) dx = \left[x^2 - \frac{x^2}{4} \right]_0^2 = 3.$$

So

$$\bar{x} = \frac{1}{A} \int_0^2 x \left(2x - \frac{x}{2} \right) dx = \frac{1}{3} \int_0^2 \left(2x^2 - \frac{x^2}{2} \right) dx = \frac{1}{3} \left[\frac{2}{3}x^3 - \frac{x^3}{6} \right]_0^2 = \frac{4}{3}$$

$$\bar{y} = \frac{1}{A} \int_0^2 (2x)^2 - \left(\frac{x}{2} \right)^2 dx = \frac{1}{3} \int_0^2 \frac{15x^2}{8} dx = \frac{1}{3} \left[\frac{5x^3}{8} \right]_0^2 = \frac{5}{3}.$$

So the centroid of the region is $(4/3, 5/3)$.

5) Let a be a constant such that $-1 < a < 0$. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{a^n(n+a)^2}$$

in terms of a .

Answer: We can use the ratio test to determine the radius of convergence:

$$\lim_{n \rightarrow \infty} \left| \frac{\frac{(x-2)^{n+1}}{a^{n+1}(n+1+a)^2}}{\frac{(x-2)^n}{a^n(n+a)^2}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x-2}{a} \right| \frac{(n+1+a)^2}{(n+a)^2} = \left| \frac{x-2}{a} \right|$$

so the power series converges when $\left| \frac{x-2}{a} \right| < 1$ that is, when $|x-2| < |a|$. Thus the interval of convergence contains the open interval $(2-|a|, 2+|a|)$ and since a is negative this is just $(2+a, 2-a)$. We now check the endpoints. At $x = 2-a$ we see that we have the series

$$\sum_{n=0}^{\infty} \frac{(-a)^n}{a^n(n+a)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+a)^2}$$

which we see is convergent by the alternating series test. At $x = 2+a$ we see that we have the series

$$\sum_{n=1}^{\infty} \frac{a^n}{a^n(n+a)^2} = \sum_{n=1}^{\infty} \frac{1}{(n+a)^2}$$

Since this series looks like $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is convergent by the p -series test, we use the limit comparison test

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{(n+a)^2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \left(\frac{n}{n+a} \right)^2 = 1$$

to get that $\sum_{n=1}^{\infty} \frac{1}{(n+a)^2}$ is also convergent. Thus the interval of convergence for the power series is $[2+a, 2-a]$.

6) Evaluate the indefinite integral as a power series. What is the radius of convergence?

$$\int \frac{1}{t^2} \left(\frac{1}{1-t^4} - 1 \right) dt.$$

Answer: Since $\frac{1}{1-t} = \sum_{n=0}^{\infty} t^n$ exactly when $|t| < 1$ it follows that $\frac{1}{1-t^4} = \sum_{n=0}^{\infty} t^{4n}$ exactly when $|t^4| < 1$, that is when $|t| < 1$. So

$$\frac{1}{1-t^4} - 1 = (1+t^4+t^8+t^{12}+\dots) - 1 = (t^4+t^8+t^{12}+\dots) = \sum_{n=1}^{\infty} t^{4n}$$

exactly when $|t| < 1$ and so

$$\frac{1}{t^2} \left(\frac{1}{1-t^4} - 1 \right) = \frac{1}{t^2} (t^4+t^8+t^{12}+\dots) = (t^2+t^6+t^{10}+\dots) = \sum_{n=1}^{\infty} t^{4n-2}$$

exactly when $|t| < 1$. Since integrating does not change the radius of convergence, we have that

$$\int \frac{1}{t^2} \left(\frac{1}{1-t^4} - 1 \right) dt = \int \sum_{n=1}^{\infty} t^{4n-2} dt = \sum_{n=1}^{\infty} \int t^{4n-2} dt = C + \sum_{n=1}^{\infty} \frac{t^{4n-1}}{4n-1}$$

where C is an arbitrary constant for $|t| < 1$.

7) Evaluate $\int x^3 \sqrt{4-x^2} dx$.

Answer: Let $x = 2 \sin \theta$, then $dx = 2 \cos \theta d\theta$ so we have

$$\int x^3 \sqrt{4-x^2} dx = \int (2 \sin \theta)^3 \sqrt{4 - (2 \sin \theta)^2} (2 \cos \theta) d\theta$$

since $\sqrt{4 - (2 \sin \theta)^2} = \sqrt{4(1 - \sin^2 \theta)} = \sqrt{4 \cos^2 \theta} = 2 \cos \theta$ we have

$$\begin{aligned} \int x^3 \sqrt{4-x^2} dx &= 2^5 \int \sin^3 \theta \cos^2 \theta d\theta = 32 \int \sin \theta \sin^2 \theta \cos^2 \theta d\theta = \\ &= 32 \int \sin \theta (1 - \cos^2 \theta) \cos^2 \theta d\theta = 32 \int (\cos^2 \theta - \cos^4 \theta) \sin \theta d\theta \end{aligned}$$

let $u = \cos \theta$ then $du = -\sin \theta d\theta$ so

$$= -32 \int (u^2 - u^4) du = -32 \frac{u^3}{3} + 32 \frac{u^5}{5} + C = -32 \frac{\cos^3 \theta}{3} + 32 \frac{\cos^5 \theta}{5} + C$$

and by drawing a triangle we can show that if $\sin \theta = x/2$ then $\cos \theta = \sqrt{4-x^2}/2$ so

$$\int x^3 \sqrt{4-x^2} dx = -32 \frac{(4-x^2)^{3/2}}{24} + 32 \frac{(4-x^2)^{5/2}}{160} + C = -\frac{4}{3}(4-x^2)^{3/2} + \frac{1}{5}(4-x^2)^{5/2} + C$$

and we can use differentiation to check that this is correct.

8) Find the implicit solution to the differential equation that satisfies the given initial condition:

$$y^2 \frac{dy}{dx} = (x^2 + 6)e^y, \quad y(1) = 0$$

Answer: First we must separate the variables to get

$$\int y^2 e^{-y} dy = \int (x^2 + 6) dx$$

the right-hand-side is relatively easy

$$\int (x^2 + 6) dx = \frac{x^3}{3} + 6x + C$$

the left-hand-side requires us to integrate by parts twice, first letting $u = y^2$ and $dv = e^{-y} dy$ we have

$$\int y^2 e^{-y} dy = -y^2 e^{-y} - \int (-e^{-y})(2y) dy = -y^2 e^{-y} + 2 \int y e^{-y} dy$$

then letting $u = y$ and $dv = e^{-y} dy$ to get

$$\int y e^{-y} dy = -y e^{-y} - \int (-e^{-y}) dy = -y e^{-y} - e^{-y} + C$$

thus

$$\int y^2 e^{-y} dy = -y^2 e^{-y} - 2y e^{-y} - 2e^{-y} + C = -e^{-y}(y^2 + 2y + 2) + C$$

so we have the general differential equation solution

$$-e^{-y}(y^2 + 2y + 2) = \frac{x^3}{3} + 6x + C$$

since $y(1) = 0$ we have

$$-e^{-0}(0^2 + 2 \cdot 0 + 2) = \frac{1}{3} + 6 + C \Rightarrow -2 = \frac{19}{3} + C \Rightarrow C = -\frac{25}{3}$$

so we have the solution

$$-e^{-y}(y^2 + 2y + 2) = \frac{x^3}{3} + 6x - \frac{25}{3}.$$

9) Evaluate

$$\int \frac{x^3 + 7x^2 + 20x + 28}{(x+2)^2(x^2+4)} dx$$

Answer: We have to use partial fractions to evaluate this, first we note that the degree of the numerator is less than the degree of the denominator, so we don't need to reduce the fraction with polynomial long division. We proceed:

$$\frac{x^3 + 7x^2 + 20x + 28}{(x+2)^2(x^2+4)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x^2+4}$$

so

$$x^3 + 7x^2 + 20x + 28 = A(x+2)(x^2+4) + B(x^2+4) + C(x+2)^2.$$

So

$$\begin{aligned} x^3 + 7x^2 + 20x + 28 &= A(x^3 + 2x^2 + 4x + 8) + B(x^2 + 4) + C(x^2 + 2x + 4) + D(x^3 + 2x^2 + 4x) \\ &= (A + D)x^3 + (2A + B + C + 2D)x^2 + (4A + 2C + 4D)x + (8A + 4B + 4C) \end{aligned}$$

So immediately $A + D = 1$. We also see that at $x = -2$ we have

$$(-2)^3 + 7(-2)^2 + 20(-2) + 28 = B((-2)^2 + 4) \Rightarrow 8 = 8B \Rightarrow B = 1.$$

Using this, we see that the constant terms of the right-hand-side are $8A + 4B + 4C = 4 + 4(2A + C) = 28$, so $2A + C = 6$. Since $2A + B + C + 2D = 2A + 1 + C + 2D = 7$ we have $2A + C + 2(1 - A) = 6$ so $C + 2 = 6$ so $C = 4$. Thus from $2A + C = 6$ we know $A = 1$ and from $A + D = 1$ we know $D = 0$.

$$\frac{x^3 + 7x^2 + 20x + 28}{(x+2)^2(x^2+4)} = \frac{1}{x+2} + \frac{1}{(x+2)^2} + \frac{4}{x^2+4}$$

so

$$\begin{aligned} \int \frac{x^3 + 7x^2 + 20x + 28}{(x+2)^2(x^2+4)} dx &= \int \frac{1}{x+2} dx + \int \frac{1}{(x+2)^2} dx + 4 \int \frac{1}{x^2+4} dx \\ &= \ln|x+2| - \frac{1}{x+2} + 2 \arctan(x/2) + C \end{aligned}$$

which we can check by differentiation.

10) Give the Taylor series expansion of $f(x) = e^x$ centered at $a = 1$. (Don't worry about $R_n(x)$ or the radius of convergence.)

Answer: We see that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!} (x-1)^n = \sum_{n=0}^{\infty} \frac{e}{n!} (x-1)^n.$$