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Math 0100: Introductory Calculus II
Practice Midterm II- Solutions

The following are the solutions to the practice midterm. It is recommended that you attempt the practice before looking at the solutions.

1) The function we are estimating the integral of is $f(x) = x^4 + x$. If $n = 4$ and we're looking at the interval $-2 \le x \le 2$, then $\Delta x = (2 - (-2))/4 = 1$. So our estimate is

$$\frac{\Delta x}{3}(f(-2) + 4f(-1) + 2f(0) + 4f(1) + f(2)) = \frac{1}{3}(14 + 0 + 0 + 8 + 18) = \frac{40}{3}.$$

We see that $f'(x) = 4x^3 + 1$ so $f''(x) = 12x^2$ so $f^{(3)}(x) = 24x$ so $f^{(4)}(x) = 24$. Thus $K = 24 \ge |f^{(4)}(x)|$ when $-2 \le x \le 2$. Using the formula for the bound:

$$|E_S| \le \frac{K(b-a)^5}{180n^4} = \frac{24(4)^5}{180(4)^4} = \frac{(24)(4)}{180} = \frac{24}{45} = \frac{8}{15}.$$

So our bound for the error is 8/15.

2) We see that

$$\frac{x + \sqrt{x}}{x^{3/2}} \ge \frac{\sqrt{x}}{x^{3/2}} = \frac{1}{x} \quad \text{when} \quad x \ge 1$$

so since $\int_1^\infty \frac{dx}{x}$ is divergent (by remembering results for $\int \frac{dx}{x^p}$ or we can check that it is divergent) the comparison test tells us that

$$\int_1^\infty \frac{x + \sqrt{x}}{x^{3/2}}\, dx$$

is divergent. A more direct way of seeing this is by noting that

$$\int_1^\infty \frac{x + \sqrt{x}}{x^{3/2}}\, dx = \lim_{t \to \infty} \int_1^t \frac{x + \sqrt{x}}{x^{3/2}}\, dx = \lim_{t \to \infty} \int_1^t (x^{-1/2} + x^{-1})\, dx$$

$$= \lim_{t \to \infty} [2\sqrt{x} + \ln|x|]_1^t = \lim_{t \to \infty} (2\sqrt{t} + \ln|t| - 2) = \infty + \infty - 2.$$

3) Since we're given the range $0 \le y \le \pi/4$, it's implied that we want to solve for x in terms of y and then differentiate. So

$$e^x = \sec x \;\Rightarrow\; x = \ln(\sec x) \;\Rightarrow\; \frac{dx}{dy} = \frac{1}{\sec y}\sec y \tan y = \tan y \;\Rightarrow\; \left(\frac{dx}{dy}\right)^2 = \tan^2 y$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + \tan^2 y = \sec^2 y.$$

So our arclength formula gives us

$$L = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^{\pi/4} \sqrt{\sec^2 y}\, dy = \int_0^{\pi/4} \sec y\, dy = [\ln|\sec y + \tan y|]_0^{\pi/4} = \ln|\sqrt{2} + 1|.$$

4) If we graph the region, we see that it is symmetric about the x-axis so $\bar{y} = 0$. The area of the bounded region is

$$A = \int_0^2 x^2 \; dx - \int_0^2 -x^2 \; dx = 2\int_0^2 x^2 \; dx = \frac{16}{3}.$$

So

$$\bar{x} = \frac{1}{A}\int_0^2 x(x^2) - x(-x^2) \; dx = \frac{3}{16}\int_0^2 2x^3 \; dx = \frac{3}{2}$$

and thus the centroid is $(3/2, 0)$.

5) The equation was separable, so separating we can write this equation in the differential form as

$$-\frac{1}{y} \; dy = (1+x) \; dx$$

and so we integrate

$$-\int y^{-1} \; dy = \int (1+x) \; dx \;\; \Rightarrow \;\; -\ln|y| = x + \frac{x^2}{2} + C.$$

Since the problem ask for an explicit solution, we solve for y by exponentiating both sides

$$|y|^{-1} = e^{x+\frac{1}{2}x^2+C} = Ae^{x+\frac{x^2}{2}}, \quad A > 0$$

so

$$y = \pm\frac{1}{A}e^{-x-\frac{x^2}{2}} = Be^{-x-\frac{x^2}{2}}$$

where B is some arbitrary nonzero constant, and since $y = 0$ is also a solution to the differential equation, B can also be zero. So a general solution to the differential equation is

$$y = Be^{-x-\frac{x^2}{2}}$$

where B is any arbitrary constant. Solving for the initial condition $y(0) = 2$ we see $y(0) = Be^0 = B = 2$ so $B = 2$ and we have the solution

$$y = 2e^{-x-\frac{x^2}{2}}.$$

6) See Example 7 in section 7.8. Short answer: it's divergent.