Mathematics 0100 Practice Final Exam Solution

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1. Find the radius of convergence and interval of convergence for the series

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)} \left(-\frac{x}{2}\right)^n$$

Solution Use the ratio test

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{1}{n+2}\left(-\frac{x}{2}\right)^{n+1} \frac{n+1}{1}\left(-\frac{2}{x}\right)^n\right| \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} \left|\frac{x}{2}\right| = \left|\frac{x}{2}\right|
\end{aligned}$$

If $\left|\frac{x}{2}\right| < 1$, $|x| < 2$, the series is convergent, If $\left|\frac{x}{2}\right| > 1$, $|x| > 2$, the series is divergent. Therefore the radius of convergence is 2.
If $x = 2$, we have an alternating harmonic series, which is convergent by the alternating series test; if $x = -2$, we have a harmonic series, which is divergent. Therefore the interval of convergence is $(-2, 2]$

2. Express the function as the sum of power series. Find the radius of convergence.

$$f(x) = \frac{3}{x^2 - x - 2}$$

Solution We express $f(x)$ into a sum of partial fractions

$$f(x) = \frac{1}{x-2} - \frac{1}{x+1}$$

Then we expand $f(x)$ into power series, we have

$$\begin{aligned}
f(x) &= -\frac{1}{2}\frac{1}{1-\frac{x}{2}} - \frac{1}{1-(-x)} \\
&= \sum_{n=0}^{\infty} (-\frac{1}{2^{n+1}} - (-1)^n)x^n
\end{aligned}$$

By the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{(\frac{1}{2^{n+2}}+(-1)^{n+1})x^{n+1}}{(\frac{1}{2^{n+1}}+(-1)^n)x^n} = |x|$$

If $|x| < 1$, it is convergent, if $|x| > 1$, it is divergent, hence the interval of convergence is $(-1, 1)$.

3. Use power series to evaluate the limit

$$\lim_{x\to 0}\frac{x-\tan^{-1}x}{x^3}$$

Solution

$$\begin{aligned} tan^{-1}x &= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+... \\ \lim_{x\to 0}\frac{x-\tan^{-1}x}{x^3} &= \frac{x-(x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...)}{x^3} \\ &= \lim_{x\to 0}\frac{\frac{x^3}{3}-\frac{x^5}{5}+\frac{x^7}{7}+...}{x^3} \\ &= \lim_{x\to 0}\frac{1}{3}-\frac{1}{5}x^2+\frac{1}{7}x^4+... = \frac{1}{3} \end{aligned}$$

4. Determine whether the series is convergent or divergent. If it is convergent, find the sum

$$\sum_{n=1}^{\infty}\left(\frac{1}{e^n}+\frac{1}{n(n+1)}\right)$$

Solution We split the series into two parts $\sum_{n=1}^{\infty}\frac{1}{e^n}$ and $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$. The first part is a geometric series with $r=\frac{1}{e}<1$, hence it is convergent with limit $\frac{\frac{1}{e}}{1-\frac{1}{e}}=\frac{1}{e-1}$. The second part is a telescoping series. Let

$$\begin{aligned} s_n &= a_1+...+a_n \\ &= \frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+...+\frac{1}{n(n+1)} = 1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+...+\frac{1}{n}-\frac{1}{n+1} \\ &= 1-\frac{1}{n+1} = \frac{n}{n+1} \end{aligned}$$

Then the second part has limit $\lim_{n\to\infty}s_n = 1$. Therefore $\lim_{n\to\infty}\frac{1}{e^n}+\frac{1}{n(n+1)}=\frac{1}{e-1}+1=\frac{e}{e-1}$

5. Solve the differential equation

$$\frac{dy}{d\theta}=\frac{e^y\sin^2\theta}{y\sec\theta}$$

Solution If we separate the variables, we have

$$ye^{-y}dy=\frac{\sin^2\theta}{\sec\theta}d\theta$$

, then

$$\int ye^{-y}dy = \int \sin^2\theta\cos\theta d\theta$$

For $\int ye^{-y}dy$, use integral by parts, let $u = y, dv = e^{-y}dy$, then $v = -e^{-y}$, we have

$$\int ye^{-y}dy = \int udv = uv - \int vdu = -ye^{-y} - \int(-e^{-y})dy = -ye^{-y} + \int e^{-y}dy = -ye^{-y} - e^{-y} + C$$

For

$$\int \sin^2\theta\cos\theta d\theta$$

, make a substitution $t = \sin\theta$, then $dt = \cos\theta d\theta$, hence

$$\int sin^2\theta d\theta = \int t^2 dt = \frac{1}{3}t^3 = \frac{1}{3}\sin^3\theta + C,$$

then the general solution for the differential equation is

$$-ye^{-y} - e^{-y} = \frac{1}{3}\sin^3\theta + C$$

6. Find the length of the arc of the parabola $y = e^x$ from (0,1) to (1,e).

Solution According to the formula of Arc length,

$$\begin{aligned} L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)} dx \\ &= \int_0^1 \sqrt{1 + e^{2x}} dx. \end{aligned}$$

Let $u = e^x$, then $du = e^x dx = udx$. Then

$$L = \int_1^e \frac{\sqrt{(1+u^2)}}{u} du$$

Assume $u = \tan\theta$, then $\sqrt{1+u^2} = |\sec\theta|$, hence

$$\begin{aligned} L &= \int_{\frac{\pi}{4}}^{\arctan e} \frac{\sec\theta}{\tan\theta}\sec^2\theta d\theta \\ &= \int_{\frac{\pi}{4}}^{\arctan e} \frac{\sec^3\theta\tan\theta}{\tan^2\theta} d\theta \end{aligned}$$

Assume $x = \sec\theta$, $dx = \sec\theta\tan\theta d\theta$, $\tan^2\theta = x^2 - 1$, then

$$\begin{aligned} L &= \int_{\sqrt{2}}^{\sqrt{e^2+1}} \frac{x^2 dx}{x^2 - 1} \\ &= \int_{\sqrt{2}}^{\sqrt{e^2+1}} dx + \int_{\sqrt{2}}^{\sqrt{e^2+1}} \frac{dx}{x^2-1} \\ &= \sqrt{e^2+1} - \sqrt{2} + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|\Bigg|_{\sqrt{2}}^{\sqrt{e^2+1}} \\ &= \sqrt{e^2+1} - \sqrt{2} - 1 + \ln(\sqrt{e^2+1} - 1) - \ln(\sqrt{2} - 1) \end{aligned}$$

7. Determine whether the integral is convergent or divergent

$$\int_0^\infty \frac{x}{(x^2+2)^2}dx$$

Solution Let $u = x^2 + 2$, then $du = 2xdx$

$$\begin{aligned}
\int_0^\infty \frac{x}{(x^2+2)^2}dx &= \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+2)^2}dx \\
&= \frac{1}{2}\lim_{t\to\infty}\int_2^{t^2+2}\frac{du}{u^2} \\
&= \frac{1}{2}\lim_{s\to\infty}\int_2^{s}\frac{du}{u^2} \\
&= \frac{1}{2}\lim_{s\to\infty}\left(\frac{1}{2}-\frac{1}{s}\right) \\
&= \frac{1}{4}
\end{aligned}$$

8. Evaluate the integral

$$\int_0^1 \frac{1}{1+\sqrt{y}}dy$$

Solution Make a substitution $u = \sqrt{y}$, then $y = u^2$, $dy = 2udu$, $y = 0, u = 0$, $y = 1, u = 1$

$$\begin{aligned}
\int_0^1 \frac{1}{1+\sqrt{(y)}}dy = \int_0^1 \frac{2u}{1+u}du \\
&= \int_0^1 2du - 2\int_0^1 \frac{1}{1+u}du \\
&= 2 - 2\ln(1+u)|_0^1 \\
&= 2 - 2\ln 2
\end{aligned}$$

9. Find the values of p for which the following series is convergent

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^p}$$

Solution According to the integral test, we only need to consider the following improper integral

$$\int_2^\infty \frac{dx}{x(\ln x)^p} = \lim_{t\to\infty}\int_2^t \frac{dx}{x(\ln x)^p}$$

Assume $y = \ln x$, then $dy = \frac{1}{x}dx$. If $x = 2, y = \ln 2$, if $x = t, y = \ln t$. Then

$$\int_2^t \frac{dx}{x(\ln x)^p} = \int_{\ln 2}^{\ln t}\frac{dy}{y^p}$$

If $p \neq 1$, we have

$$\int_{\ln 2}^{\ln t} y^{-p} dy = \frac{(\ln t)^{1-p} - (\ln 2)^{1-p}}{1-p}$$

If $p = 1$, we have

$$\int_{\ln 2}^{\ln t} \frac{dy}{y} = lnlnt - lnln2$$

. If $p > 1$,

$$\lim_{t\to\infty} \int_2^t \frac{dx}{x(\ln x)^p} = \frac{1}{(p-1)(\ln 2)^{p-1}} < \infty$$

If $p = 1$

$$\lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{dy}{y} = \lim_{t\to\infty} lnlnt - lnln2 = \infty$$

If $p < 1$

$$\lim_{t\to\infty} \int_2^t \frac{dx}{x(\ln x)^p} = \lim_{t\to\infty} \frac{(\ln t)^{1-p} - (\ln 2)^{1-p}}{1-p} = \infty$$

Since $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is convergent if and only if the improper integral

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p}$$

is convergent. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is convergent if and only if $p > 1$

10. If $\sum a_n$ is a convergent series with positive terms, is it true that

$$\sum [e^{a_n} - 1]$$

is also convergent?

Solution Because $\sum a_n$ is convergent, test of divergence says that

$$\lim_{n\to\infty} a_n = 0$$

. By the continuity of the function e^{a_n}

$$\lim_{n\to\infty} (e^{a_n} - 1) = e^{\lim_{n\to\infty} a_n} - 1 = 0$$

According to the l'Hospital's Rule

$$\begin{aligned} \lim_{n\to\infty} \frac{e^{a_n} - 1}{a_n} &= \lim_{x\to 0} \frac{e^x - 1}{x} \\ &= \lim_{x\to 0} \frac{(e^x - 1)'}{x'} = \lim_{x\to 0} e^x = 1 \end{aligned}$$

Then the limit comparison test says that if $\sum a_n$ is convergent, $\sum [e^{a_n} - 1]$ is convergent.