SAMPLE SOLUTIONS

1. Solution: Note the power of $\sec x$ is even, we use the substitution $u = \tan x$, then $du = \sec^2 x dx$, hence

$$\int \tan x \sec^4 x dx = \int u(1+u^2)du = \frac{1}{2}u^2 + \frac{1}{4}u^4 + C = \frac{1}{2}\tan^2 x + \frac{1}{4}\tan^4 x + C$$

Remark 0.1. Recall that $d(\tan x) = \sec^2 x dx$, $d(\sec x) = \sec x \tan x dx$, $\sec^2 x = 1 + \tan^2 x$.

2. Solution: Remember the formula for the arc length,

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Since

$$y'(x) = \frac{x^2}{2} - \frac{1}{2x^2}$$

the length of the curve is

$$\int_1^2 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\, dx = \int_1^2 \sqrt{\left(\frac{x^2}{2} + \frac{1}{2x^2}\right)^2}\, dx$$
$$= \int_1^2 \frac{x^2}{2} + \frac{1}{2x^2}\, dx = \frac{1}{6}x^3 - \frac{1}{2x}\Big]_1^2 = \frac{17}{12}$$

3. Solution: This is separable equation (the only required equation for the differential equation chapter), so the first step will always be 'separate the equation': (recall we tend to write y' as $\frac{dy}{dx}$)

$$\frac{dy}{y(y+1)} = xdx$$

The next step is integrate both sides,

$$\int \frac{1}{y(y+1)}, dy = \int \frac{1}{y} - \frac{1}{y+1} dy = \ln|y| - \ln|y+1| = \frac{1}{2}x^2 + C$$

from which we get

$$\frac{y}{y+1} = Ae^{\frac{1}{2}x^2}$$

use the initial condition we get $\frac{1}{2} = Ae^0$, i.e. $A = \frac{1}{2}$, hence the solution to the differential equation is

$$y = \frac{1}{1 - \frac{1}{2}e^{-\frac{1}{2}x^2}}$$

4. Solution: We use $|\sin^3 n| \leq 1$ here, then comparison test: since

$$\left| \left(\frac{\sin n}{n} \right)^3 \right| \leq \frac{1}{n^3}$$

$\sum 1/n^3$ is convergent since it is p-series with $p = 3$, we have the original series is absolutely convergent by comparison test and the definition of absolutely convergence.

Remark 0.2. Even you are only ask to prove the convergence of this series, you still have to use absolute value here, because the comparison test is for series with positive terms. Then use absolutely convergence implies convergence.

5. Solution: Here we use integral test, (mostly motivated by the appearance of $\ln n$ and $1/n$), Since

$$\int_2^\infty \frac{1}{x(\ln x)^2}\, dx = \int_{ln2}^\infty \frac{1}{u^2}\, du$$

if we use substitution $u = \ln x$, but the latter integral equals

$$\lim_{t\to\infty} \int_{\ln 2}^t \frac{1}{u^2}\, du = \lim_{t\to\infty} -\frac{1}{u}\bigg]_{\ln 2}^t = \frac{1}{\ln 2}$$

which is finite, which in turn implies the series $\sum_{n=2}^\infty \frac{1}{n(\ln n)^2}$ is convergent.

6. Solution: Note the series is power series with coefficients $0, 1, 0, 0, 1, 0, 0, 0, 0, 1, \cdots$, which motivate us to use the ratio test,

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{(n+1)^2}}{x^{n^2}} \right| = \left| x^{2n+1} \right|$$

whose limit will be 0 if $|x| < 1$, and $0 < 1$, hence by ratio test, the series is convergent. If $|x| \geq 1$, the limit of x^{n^2} will not exist unless $x = 1$, in which case the limit is 1 which is not 0, by test for divergence, we know the series diverges for these values of x. Thus the interval of convergence is $(-1, 1)$.

7. Solution: We know the power series expansion for $1/(1 + x)$, while $\ln(1 + x)$ is the antiderivative of that, so

$$\ln(1+x) = \int \frac{1}{1+x}\, dx = \int \frac{1}{1-(-x)}\, dx = \int \sum_{n=0}^\infty (-1)^n x^n\, dx = \sum_{n=0}^\infty (-1)^n \frac{x^{n+1}}{n+1}$$

We know the radius of convergence for this series is still 1, since the radius of convergence of $\sum(-x)^n$ is(or we can just deduce this by ratio test). For the endpoints, if $x = 1$, the series is convergent by alternating series test, if $x = -1$, the series is divergent since it is then harmonic series. Thus the interval of convergence is $(-1, 1]$.

8. Solution: We need to memorize the Maclaurin series for $\sin x$, then plug in $2x^2$ in the position of x, then divide each term by x, we get

$$\frac{1}{x}\sin(2x^2) = \frac{1}{x}\sum_{n=0}^\infty (-1)^n \frac{(2x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^\infty (-1)^n \frac{2^{2n+1}}{(2n+1)!} x^{4n+1}$$

9. Solution: We know the Maclaurin series for e^x is $\sum_{n=0}^{\infty} x^n/n!$, set $x = -1/10$ we get

$$e^{-1/10} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{10^n n!}$$

Note it is alternating series, and

$$\frac{1}{10^3 \cdot 6} < 0.01$$

hence

$$e^{-1/10} \approx 1 - \frac{1}{10} = \frac{9}{10}$$

with error less than 0.01

10. Solution:

$$\begin{aligned} f(x) &= \frac{1}{x}, \quad f(-1) = -1 \\ f'(x) &= -\frac{1}{x^2}, \quad f'(-1) = -1 \\ f''(x) &= \frac{2}{x^3}, \quad f''(-1) = -2 \\ f'''(x) &= -\frac{2 \cdot 3}{x^4}, \quad f'''(-1) = -2 \cdot 3 \end{aligned}$$

we get from the pattern here that $f^{(n)}(-1) = -n!$, hence the Taylor series for $1/x$ at -1 is

$$\sum_{n=0}^{\infty} \frac{-n!}{n!}(x+1)^n = \sum_{n=0}^{\infty} -(x+1)^n$$