HOMEWORK 11 SOLUTIONS

P. 733: 6, 10, 12, 14, 24, 32

P. 746: 8, 10, 18, 22, 34, 56, 62

SECTION 11.9

(6) $$f(x) = \frac{1}{x+10} = \frac{1}{10}\frac{1}{1+\frac{x}{10}} = \frac{1}{10}\left(\underbrace{\frac{1}{1-\left(-\frac{x}{10}\right)}}_{\text{has a geometric series rep.}}\right) = \frac{1}{10}\sum_{n=0}^{\infty}\left(-\frac{x}{10}\right)^n$$

$$= \frac{1}{10}\sum_{n=0}^{\infty}\frac{(-1)^n}{10^n}x^n = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n}{10^{n+1}}x^n}$$

converges when $\left|-\frac{x}{10}\right| < 1$ so $|x| < 10$

So using the interval of convergence for a geometric series we have the interval of convergence $\boxed{(-10, 10)}$.

(10) $$f(x) = \frac{x^2}{a^3 - x^3} = \frac{x^2}{a^3}\left(\underbrace{\frac{1}{1-\left(\frac{x}{a}\right)^3}}_{\text{has a geometric series rep.}}\right) = \frac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\left(\frac{x}{a}\right)^3\right)^n = \sum_{n=0}^{\infty}\frac{x^2}{a^3}\frac{x^{3n}}{a^{3n}}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{x^{3n+2}}{a^{3n+3}}}$$

converges when $\left|\left(\frac{x}{a}\right)^3\right| < 1$

$\Downarrow$

$|x^3| < |a^3|$

$\Downarrow$

$|x| < |a|$

The interval of convergence is $(-|a|, |a|)$.

(12) $f(x) = \dfrac{x+2}{2x^2-x-1} = \dfrac{x+2}{(2x+1)(x-1)} = \dfrac{A}{2x+1} + \dfrac{B}{x-1}$

So $\quad x+2 = A(x-1) + B(2x+1)$

So $\quad x+2 = (A+2B)x + (-A+B)$

So $\quad A+2B = 1$
$\quad -A+B = 2$ Adding these

$(A+2B)+(-A+B) = 1+2$

$\Downarrow$

$3B = 3$

$B = 1$

So $A = -1$ so

$f(x) = \dfrac{x+2}{2x^2-x-1} = \dfrac{-1}{2x+1} + \dfrac{1}{x-1} = -\left(\dfrac{1}{1-(-2x)}\right) - \left(\dfrac{1}{1-x}\right)$

Both have geometric series representations

$= -\sum_{n=0}^{\infty}(-2x)^n - \sum_{n=0}^{\infty}x^n = -\sum_{n=0}^{\infty}(-2)^n x^n - \sum_{n=0}^{\infty}x^n$ when $|x| < \frac{1}{2}$

converges when $|-2x|<1$ $\Downarrow$ $\underbrace{|x| < \tfrac{1}{2}}_{\text{smaller radius of convergence}}$

converges when $\Downarrow$ $|x|<1$

So the interval of convergence is $(-\frac{1}{2}, \frac{1}{2})$.

(14) a) FROM EXAMPLE 6 in this section we know ← worth Recalling!

$\ln(1-x) = -\sum_{n=1}^{\infty}\dfrac{x^n}{n}$ when $|x|<1$

So $\ln(1+x) = \ln(1-(-x)) = -\sum_{n=1}^{\infty}\dfrac{(-x)^n}{n} = -\sum_{n=1}^{\infty}(-1)^n\dfrac{x^n}{n} = \boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{x^n}{n}}$ when $|-x|<1$

$\Rightarrow$ when $|x|<1$

Radius is still 1.

(14) b) $x\ln(1+x) = x\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = \boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{n+1}}{n}}$

c) $\ln(1+x^2) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x^2)^n}{n} = \boxed{\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n}}{n}}$

(24) From example 6 $\ln(1-t) = -\sum_{n=1}^{\infty}\frac{t^n}{n}$, $|t|<1 \Rightarrow R=1$

So $\frac{1}{t}\ln(1-t) = -\frac{1}{t}\sum_{n=1}^{\infty}\frac{t^n}{n} = -\sum_{n=1}^{\infty}\frac{t^{n-1}}{n} = -1-\frac{t}{2}-\frac{t^2}{3}-\frac{t^3}{4}-\dots$

Multiplying by $\frac{1}{t}$ does not change the radius of convergence, and the series is well-defined when $t=0$. So

$$\int\frac{1}{t}\ln(1-t)\,dt = -\int\left(\sum_{n=1}^{\infty}\frac{t^{n-1}}{n}\right)dt = -\sum_{n=1}^{\infty}\int\frac{t^{n-1}}{n}\,dt = \boxed{-\sum_{n=1}^{\infty}\frac{t^n}{n^2}+C}$$

Radius of convergence is unchanged by power series

So $\boxed{R=1}$

(32) If $f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ then $f'(x) = \sum_{n=0}^{\infty}\frac{(-1)^n(2n)x^{2n-1}}{(2n)!}$

and $f''(x) = \sum_{n=0}^{\infty}\frac{(-1)^n(2n)(2n-1)x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty}\frac{(-1)^n(2n)(2n-1)x^{2n-2}}{(2n)!}$ since the first term is 0.

Since $(2n)! = (2n)(2n-1)\underbrace{(2n-2)(2n-3)\cdots 3\cdot 2\cdot 1}_{(2n-2)!} = (2n)(2n-1)(2n-2)!$

we have $\frac{(2n)(2n-1)}{(2n)!} = \frac{1}{(2n-2)!}$. So $f''(x) = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2(n-1)}}{(2(n-1))!}$

$$= \sum_{\underbrace{n-1}_{k}=0}^{\infty}\frac{(-1)^{\overset{k+1}{n}}x^{2\underbrace{(n-1)}_{k}}}{(2\underbrace{(n-1)}_{k})!} = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}x^{2k}}{(2k)!} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n)!} = -\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$= -f(x)$$

So $f(x)+f''(x) = f(x)+(-f(x)) = 0$.

(8) $f(x) = \cos 3x$
$f'(x) = -3 \sin 3x$
$f''(x) = -9 \cos 3x$
$f'''(x) = 27 \sin 3x$
$f^{(4)}(x) = 81 \cos 3x$
$\vdots$
$f^{(4n)}(x) = 3^{4n} \cos 3x$
$f^{(4n+1)}(x) = -3^{4n+1} \sin 3x$
$f^{(4n+2)}(x) = -3^{4n+2} \cos 3x$
$f^{(4n+3)}(x) = 3^{4n+3} \sin 3x$

so

$f(0) = 1$
$f'(0) = 0$
$f''(0) = -9$
$f'''(0) = 0$
$f^{(4)}(0) = 81$
$\vdots$
$f^{(4n)}(0) = 3^{4n}$
$f^{(4n+1)}(0) = 0$
$f^{(4n+2)}(0) = -3^{4n+2}$
$f^{(4n+3)}(0) = 0$

Even terms alternate, give powers of NINE.

odd terms are zero

So we have $$\boxed{\sum_{n=0}^{\infty} \underbrace{(-9)^n}_{\text{alternating powers of 9}} \frac{x^{2n}}{(2n)!}}$$

only even terms survive.

using $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

USE RATIO TEST TO GET RADIUS OF CONVERGENCE

$$\lim_{n\to\infty} \left| \frac{(-9)^{n+1} \frac{x^{2n+2}}{(2n+2)!}}{(-9)^n \frac{x^{2n}}{(2n)!}} \right| = \lim_{n\to\infty} \frac{9|x|^2}{(2n+2)(2n+1)} = 0 < 1 \quad \text{for all } x$$

So $\boxed{R = \infty}$

(10) $f(x) = xe^x$
$f'(x) = xe^x + e^x$
$f''(x) = xe^x + 2e^x$
$f'''(x) = xe^x + 3e^x$
$\vdots$
$f^{(n)}(x) = xe^x + ne^x$

so

$f(0) = 0$
$f'(0) = 1$
$f''(0) = 2$
$f'''(0) = 3$
$\vdots$
$f^{(n)}(0) = n$

So we have $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{n}{n!} x^n$ (5

$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^n = \sum_{\underbrace{n-1}_{k}=0}^{\infty} \frac{1}{\underbrace{(n-1)}_{k}!} x^{\overset{k+1}{n}} = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!} = \boxed{\sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}}$

USE RATIO TEST TO FIND RADIUS OF CONVERGENCE

$\lim_{n\to\infty} \left| \frac{x^{n+2}}{(n+1)!} \Big/ \frac{x^{n+1}}{n!} \right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$ for all x so

$\boxed{R = \infty}$

(18) $f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$f'''(x) = -\cos(x)$

$f^{(4)}(x) = \sin(x)$

So

$f(\frac{\pi}{2}) = 1$

$f'(\frac{\pi}{2}) = 0$

$f''(\frac{\pi}{2}) = -1$

$f'''(\frac{\pi}{2}) = 0$

$f^{(4)}(\frac{\pi}{2}) = 1$

only alternating, even terms survive

So $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{2})}{n!}(x - \frac{\pi}{2})^n = 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} - \dots$

$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(x-\frac{\pi}{2})^{2n}}{(2n)!}}$

(22) First we need to show the series converges for all x, for this we use the ratio test:

$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}(x-\frac{\pi}{2})^{2n+2}}{(2n+2)!} \Big/ \frac{(-1)^n (x-\frac{\pi}{2})^{2n}}{(2n)!} \right| = \lim_{n\to\infty} \frac{|x-\frac{\pi}{2}|^2}{(2n+2)(2n+1)} = 0 < 1$ for all x so $R = \infty$

So it does indeed converge for all x.

Now for arbitrary $d \geq 0$ we see that since (6)

$f^{(n+1)}(x) = \pm \cos x$ or $\pm \sin x$ for all n,

$|f^{(n+1)}(x)| \leq 1 = M$ for all x such that $|x - \frac{\pi}{2}| \leq d$

So we use Taylor's Inequality to see that

$$0 \leq |R_n(x)| \leq \frac{1}{(n+1)!} |x - \tfrac{\pi}{2}|^{n+1} \quad \text{for all } n, \text{ and } |x - \tfrac{\pi}{2}| \leq d$$

We know $\displaystyle\lim_{n\to\infty} \frac{|x - \frac{\pi}{2}|^{n+1}}{(n+1)!} = 0$ so by Squeeze theorem

$\displaystyle\lim_{n\to\infty} |R_n(x)| = 0$ so $\displaystyle\lim_{n\to\infty} R_n(x) = 0$ for all

x such that $|x - \frac{\pi}{2}| \leq d$. Since d can be made as large as we want, we have that

$\displaystyle\lim_{n\to\infty} R_n(x) = 0$ for all x, so the

Series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(x - \frac{\pi}{2})^{2n}}{(2n)!} = \sin x$ for all x,

(34) Since $\displaystyle\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $|x| < 1$

we have $\displaystyle\tan^{-1} x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{2n+1}$

So
$$\boxed{x^2 \tan^{-1} x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+5}}{2n+1}}$$

(56) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{1+x-e^x} = \lim\limits_{x\to 0} \dfrac{1-\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\dots\right)}{1+x-\left(1+x+\frac{x^2}{2}+\dots\right)} = \lim\limits_{x\to\infty} \dfrac{\frac{x^2}{2}+\dots}{-\frac{x^2}{2}+\dots}$

$= \lim\limits_{x\to\infty} \dfrac{\frac{1}{2}+\overbrace{\dots}^{\text{Powers of } x}}{-\frac{1}{2}+\underbrace{\dots}_{\text{Powers of } x}} = \boxed{-1}$.

(62) $e^x \ln(1-x) = \cancel{(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\dots)}$

$= \left(\sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}\right)\Big(\underbrace{-\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n}}_{\uparrow \text{ From Example 6}}\Big) = -\left(1+x+\frac{x^2}{2}+\dots\right)\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\dots\right)$

put the minus in at the end

$$
\begin{array}{rl}
 & 1 + x + \frac{x^2}{2} + \dots \\
= \otimes & \quad x + \frac{x^2}{2} + \frac{x^3}{3} + \dots \\
\hline
 & x + x^2 + \frac{x^3}{2} + \dots \\
+ & \qquad \frac{x^2}{2} + \frac{x^3}{2} + \dots \\
+ & \qquad\qquad \frac{x^3}{3} + \dots \\
 & \qquad\qquad \vdots \qquad \vdots \\
\hline
 & x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \dots
\end{array}
$$

So putting the minus back in our first three terms are $\boxed{-x-\frac{3}{2}x^2-\frac{4}{3}x^3}$