Spring 2015
MA 2110, Introduction to Manifolds
Supplement # 4
Classifying 1-Manifolds

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Using the method of Problem 4, we can classify all connected 1-dimensional manifolds, both topological and smooth.

Suppose first that M is a connected topological 1-manifold with $M = U \cup V$, where U and V are open and there are homeomorphisms $x : U \to \mathbb{R}$, $y : V \to \mathbb{R}$. As in the solution to Problem 4, this leads to a homeomorphism $h : x(U \cap V) \to y(U \cap V)$ between nonempty open subsets of $\mathbb{R}$. If J is any component of $U \cap V$ then each of the sets $x(J)$ and $y(J)$ is a connected open subset of $\mathbb{R}$ and therefore either a finite open interval or an open half-line or all of $\mathbb{R}$. If either of them is all of $\mathbb{R}$ then we are in the trivial case where one of U and V contains the other and therefore equals M. The Hausdorff condition on M rules out the possibility that for some J both $x(J)$ and $y(J)$ are finite intervals, and also rules out the case when one is a finite interval and the other is a half-line. This leaves only the case where both of them are half-lines. Furthermore the Hausdorff condition insures that, if both $x(J)$ and $y(J)$ are right half-lines, or if both are left half-lines, then h maps them in an order-reversing way, whereas if one is left and the other is right then the order is preserved. Examining cases, we see that either there is just one J, in which case M is homeomorphic to a line as in part (c) of the problem, or there are two J's and by a similar argument (which I will skip) M is homeomorphic to the circle.

Now more generally suppose that a connected 1-manifold M is the union of finitely many open sets $U_1, \ldots, U_r$, each homeomorphic to a line. We can arrange these in order in such a way that for each j with $1 \leq j \leq r$ the set $M_j = \cup_{1 \leq i \leq j} U_i$ is connected. Now if M_{j-1} is (homeomorphic to) a line then M_j is either a line or a circle, by the argument sketched above. If M_j is a circle then M_j is all of M, being open and (compact, therefore) closed. So, unless M is a circle, by induction each M_j is a line (including M_r, which is M).

We can go further and show that every connected 1-dimensional manifold M is homeomorphic to either $\mathbb{R}$ or S^1. It remains to consider the case where M is the union of infinitely many U_i all homeomorphic to lines. (We can assume countably infinite, because M is second countable.) Again the U_i can be arranged in a sequence so that each $M_j = \cup_{1 \leq i \leq j} U_i$ is connected. Assume M is not a circle. Then each M_j will be a line. Furthermore, as we go along, we can choose homeomorphisms $f_j : M_j \to J_j$ to finite open intervals in $\mathbb{R}$, each time choosing f_j to extend f_{j-1}. Thus in the end M is homeomorphic the union of the intervals $J_1 \subset J_2 \subset \dots$.

Using the technique of (d), this approach can also be adapted to show that every smooth 1-manifold is diffeomorphic to either the line or the circle.