Spring 2015
MA 2110, Introduction to Manifolds
Supplement #1
Smooth Maps Between Open Subsets of Euclidean Spaces

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Let $U \subset \mathbb{R}^n$ be open. A mapping $F : U \to \mathbb{R}^m$ is said to be differentiable at the point $p \in U$ if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that for $x \in U$

$$F(x) = F(p) + L(x - p) + E(x - p)$$

with the error term E satisfying $\frac{E(x-p)}{|x-p|} \to 0$ as $x \mapsto p$. In other words, for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $|x - p| < \delta$ then $|E(x - p)| < \epsilon|x - p|$.

Remark 1: One can just as well make this definition for any two finite-dimensional vector spaces instead of $\mathbb{R}^n$ and $\mathbb{R}^m$. (One could even allow the vector spaces to be infinite-dimensional, as long as they are both provided with norms. But that need not concern us here.)

Note that in particular this implies that, for any nonzero vector $v \in \mathbb{R}^n$, as the real number t approaches 0 then $\frac{E(tv)}{t|v|}$ approaches 0. It follows that

$$L(v) = lim_{t\to 0} \frac{F(p + tv) - F(p)}{t}.$$

Thus L is unique if it exists. We call L the derivative of F at p and denote it by D_pF.

Remark 2: One can make examples where the limit exists for all v but is not a linear function of v. Thus the existence of the derivative is stronger than the existence of all directional derivatives.

As a special case of the last equation, letting e_j denote the jth standard basis vector $\mathbb{R}^n$ we have

$$(D_pF)(e_j) = lim_{t\to 0} \frac{F(x_1, \ldots, x_j + t, \ldots, x_n) - F(x_1, \ldots, x_j, \ldots, x_n)}{t}.$$

In other words, writing $F(x) = (F_1(x), \ldots, F_m(x))$, the ith coordinate of the vector $(D_pF)(e_j)$ is the partial derivative

$$\frac{\partial F_i(x)}{x_j}|_{x=p}.$$

That is, the matrix corresponding to the linear map D_pF is the Jacobian matrix of partial derivatives $\frac{\partial F_i(X)}{\partial x_j}$ at the point p.

Remark 3: One can make examples in which the limit above exists when v is a standard basis vector but not for all v.

We say that F is *differentiable* if it is differentiable at every point in its domain, that is, if D_pF exists for all $p \in U$. We say that F is a C^1 map if F is differentiable and the map $DF : p \mapsto D_pF$ is continuous. (Here DF maps U to the (mn-dimensional) vector space of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.) We say that F is twice differentiable if DF exists and is differentiable, and that F is C^2 if DF is C^1, and so on: recursively F is $k+1$ times differentiable if DF exists and is k times differentiable, and F is C^{k+1} if DF exists and is C^k. We will say that F is *smooth* if it is C^∞, that is, if it is C^k for all k.

Lemma: F is C^1 if and only if for each i and j the partial derivative $\frac{\partial F_i(x)}{x_j}$ exists at each point $p \in U$ and depends continuously on p

By Remark 2 above, the Lemma is not entirely trivial.

Proof: One direction is clear. For the converse, we suppose for simplicity that $m = 1$. (Certainly F will be C^1 if each F_i is C^1.) So let f be a real-valued function on U and suppose that the functions $\frac{\partial f}{\partial x_j}$ are defined and continuous. We have to prove that f is differentiable at p. If it is, then the linear function D_pf will certainly depend continuously on p because the corresponding $1 \times n$ matrix consists of the partial derivatives, which we have assumed to be continuous. To prove that D_pf exists we may as well replace f by $x \mapsto f(x) - f(p) - L(x-p)$, where L is the linear map given by the partial derivatives at p. That is, without loss of generality the first-order partial derivatives vanish at p. Since these are continuous, for any $\epsilon > 0$ there is a neighborhood of p in which $|\frac{\partial f}{\partial x_j}| < \epsilon$ for all j. We may take the neighborhood to be given by $|x_j - p_j| < \delta$ for some $\delta > 0$. Now suppose that x is in that neighborhood. There is a path from p to x given by moving along n straight-line paths in turn, each one parallel to one of the n coordinate axes. The change in f along the jth path has absolute value bounded by $\epsilon|x_j - p_j|$. Therefore $|f(x) - f(p)| \le n\epsilon|x-p|$, and $\frac{|f(x)-f(p)|}{|x-p|} \le n\epsilon$. This proves that the derivative of f at p exists (and is zero).

Theorem (Chain Rule): When $F : U \to V$ and $G : V \to W$ are differentiable (U, V, and W being open subsets of vector spaces), then $G \circ F$ is also differentiable, and the derivative of the composed map is given by the following version of the chain rule: $D_p(G \circ F) = D_{F(p)}G \circ D_pF$.

The proof is an exercise.

It is clear that if F and G are smooth then so is $G \circ F$.

A map $F : U \to V$ between open sets in vector spaces is a *diffeomorphism* if it is smooth and has a smooth inverse $V \to U$. By the Chain Rule the derivative of a diffeomorphism is an invertible linear map. The following converse of this is a standard result, which will not be proved here:

Theorem (Inverse Function Theorem): Suppose that $F : U \to \mathbb{R}^n$ is smooth, $p \in U \subset \mathbb{R}^n$ open.

Suppose that D_pF is invertible. Then there is a possibly smaller open neighborhood $V \subset U$ of p such that $F(V)$ is open and $F|_V : V \to F(V)$ is a diffeomorphism.

In other words, a smooth map is locally invertible if and only if it is infinitesimally invertible. This is a key tool in this subject.