Spring 2015
MA 2110, Introduction to Manifolds
Supplement # 6
Families of maps and Sard's Theorem

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1 One way to use Sard's Theorem

Sometimes we want to prove that there exists a smooth map with a certain property. If the property is a transversality property, then we can often achieve this by making a smooth family of maps and showing that almost all of the maps in the family have that property. For that, we use Sard's Theorem to reduce to showing that in some sense the whole family, viewed as one map, has that property.

Here is a precise statement:

Lemma: Let Λ, P, and N be smooth manifolds and suppose that $Q \subset N$ is a smooth submanifold. Suppose that $F : \Lambda \times P \to N$ is a smooth map, and for each $\lambda \in \Lambda$ write $F_\lambda : P \to N$ for the smooth map given by $F_\lambda(a) = F(\lambda, a)$. If the map F is transverse to Q then for almost every $\lambda \in \Lambda$ the map F_λ is transverse to W.

Proof: The key is to consider the submanifold

$$Q' = F^{-1}(Q) \subset \Lambda \times P.$$

Let $\pi : Q' \to \Lambda$ be given by $(\lambda, a) \mapsto \lambda$. This is a smooth map (it is the composition

$$Q' \to \Lambda \times P \to \Lambda$$

of a projection and an inclusion).

Claim: A point $(\lambda, a) \in Q'$ is a regular point for π if and only if the map F_λ is transverse to Q at $a \in P$.

This is all that we need for the lemma: It implies that F_λ is transverse to Q (at all points in $F_\lambda^{-1}(Q)$) if and only if λ is a regular value for π, and Sard's Theorem insures that this is the case for almost every λ.

To prove the Claim, take any point $(\lambda, a) \in Q'$ and write $T_{(\lambda,a)}(\Lambda \times P)$ as the direct sum $T^{vert} \oplus T^{horiz}$ of the "vertical" and "horizontal" parts; the vertical part, isomorphic to T_aP, is the kernel of projection to $T_\lambda\Lambda$, while the horizontal part, isomorphic to $T_\lambda\Lambda$, is the kernel of projection to T_aP.

Let b be $F(\lambda, a)$. This point is in Q. Let ν be the normal space T_bN/T_bQ, and consider the linear map

$$L : T^{vert} \oplus T^{horiz} \to \nu$$

that is the composition

$$T_{(\lambda,a)}(\Lambda \times P) \to T_bN \to T_bN/T_bQ$$

of the quotient map with the derivative of F. By assumption L is surjective. Its kernel K is the tangent space $T_{(\lambda,a)}Q'$.

To say that the point (λ, a) is a regular point for π is to say that for every $h \in T^{horiz}$ there exists $v \in T^{vert}$ such that $v + h \in K$. This is equivalent to saying that for every h there exists v such that $L(v) = -L(h)$. This is equivalent to saying that $L(T^{vert} + T^{horiz}) = L(T^{vert})$. This is equivalent to saying that the restriction of L to T^{vert} is surjective. This is equivalent to saying that F_λ is transverse to Q at a.

QED

Remark: In choosing a family of maps to apply the Lemma, we often arrange for the map $\lambda \mapsto F(\lambda, a)$ to be transverse to Q for every point $a \in P$. That is, in the notation of the proof above, we choose the family F in such a way that at every point in Q' the restriction of L to T^{horiz} in surjective, where our goal is to insure that there are (in some sense) many points where the restriction of L to T^{vert} in surjective.

2 First example

In the case of two submanifolds of a vector space this immediately gives us a way of moving one of them to make it transverse to the other. Suppose that P and Q are smooth submanifolds of $\mathbb{R}^n$. Choose the parameter space (Λ in the Lemma above) to be $\mathbb{R}^n$, and consider the map $F : \mathbb{R}^n \times P \to \mathbb{R}^n$ given by $F(w, a) = w + a$. For any fixed choice of $a \in P$ the map $w \mapsto F(w, a)$ is transverse to Q (in fact, transverse to any submanifoild – it is a diffeomorphism). Therefore F itself is transverse to Q, and we can conclude that for almost every vector $w \in \mathbb{R}^n$ the map F_w is transverse to Q; in other words, for almost every w the translated manifold $w + P$ is transverse to Q.

Later we will show more generally that if P and Q are smooth submanifolds of a manifold N then P may be perturbed to make it transverse to Q.

3 Second example

We show how to produce Morse functions on a manifold P, assuming that P is a submanifold of $\mathbb{R}^n$. In fact we show that there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}$ such that the restriction of L to P has only non degenerate critical points.

In order to apply the Lemma, we must first express nondegeneracy as a transversality condition.

In the cotangent manifold T^*P let Q be the submanifold consisting of zero vectors, in other words the image of the embedding $s_0 : P \to T^*P$ that takes the point a to the vector $0_a \in T_a^*P$.

A smooth function $f : P \to \mathbb{R}$ determines a smooth map $df : P \to T^*P$. A critical point for f is any point $a \in P$ such that $d_a f = 0$, i.e. it is any point in the preimage of Q. It is a nondegenerate critical point if and only if the map $df : P \to T^*P$ is transverse to Q at a.

Now suppose that we have a smooth function $f : \Lambda \times P \to \mathbb{R}$. We may view it as a (smoothly parametrized) family of smooth functions $f_\lambda : P \to \mathbb{R}$ by writing $f_\lambda(a) = f(\lambda, a)$. As such it determines a family of smooth maps $df_\lambda : P \to T^*P$, and this does in fact correspond to a single smooth map $\Lambda \times P \to T^*P$ by writing $(\lambda, a) \mapsto d_a f_\lambda$.

If we can choose f in such a way that this map $\Lambda \times P \to T^*P$ is transverse to the manifold Q of zero vectors, then it will follow by the Lemma that for almost all λ the map df_λ is transverse to Q, that is, for almost all λ the function f_λ is a Morse function.

In the special case when P is a submanifold of $V = \mathbb{R}^n$ we can do this very simply by choosing Λ to be the vector space of all linear maps $\lambda : V \to \mathbb{R}$ and letting f_λ by the map $P \to \mathbb{R}$ given by restriction of λ. This succeeds because the resulting map $\Lambda \times P \to T^*P$ is transverse to Q. In fact for every $a \in P$ the map $\lambda \mapsto d_a f_\lambda$ is transverse to Q. (To see this, note that this map takes the vector space Λ into the submanifold T_a^*P. Identify the normal space of Q in T^*P at 0_a with T_a^*P in the obvious way, and note that our map from (the tangent space of) Λ to this normal space of Q is now simply the linear map $\Lambda \to T_a^*P$ that is dual to the linear map $T_aP \to V$. Since the dual of an injective linear map is surjective, we are done.)