March 2015
MA 2110, Introduction to Manifolds
Sixth Assignment
Due 4/2/15

March 23, 2015

We know (Problem 13) that if $(p,p) \in \Delta_M$ is a limit of double points for $F : M \to N$ then p is a non-immersion point for F. In particular, if F is an immersion then there is a neighborhood of Δ_M in $M \times M$ containing no double points.

Say that a double point (p,q) of F is a *good* double point if the map $F \times F$ is transverse to Δ_N at (p,q); otherwise call it a *bad* double point. Say that a non-immersion point p of F is a *good* non-immersion point if for every non-zero vector $v \in ker(D_pF)$ the map $DF : TM \to TN$ is transverse to 0_N at v; otherwise call it a *bad* non-immersion point. Our main goal (Problem 18 below) is to show that every smooth F (with compact domain) can be approximated by a smooth map having no bad double points and no bad non-immersion points. This includes the statement that every smooth map from an m-manifold to a $(2m+1)$-manifold can be approximated by a smooth embedding.

16. Show that if (p,p) is a limit of bad double points for F then p is a bad non-immersion point for F. It follows that if every non-immersion point of F is good then there is a neighborhood of Δ_M in $M \times M$ in which there are no bad double points.

[Hint: This is a local question, so we may assume that M is open in $\mathbb{R}^m$ and $N = \mathbb{R}^n$. Define $G(t,x,v) = \frac{F(x+tv)-F(x)}{t}$ if $t \neq 0$ and $G(x,0,v) = (D_xF)(v)$. Show that for $t \neq 0$ $G(t,x,v) = 0$ if and only if $(x, x+tv)$ is a double point for F, and that it is a good double point if and only if G is transverse to 0 at (t,x,v). Show that there exists v such that $G(0,x,v) = 0$ if and only if x is a non-immersion point for F, and that it is a good non-immersion point if and only if for every such v the map $(x,v) \mapsto G(0,x,v)$ is transverse to 0.]

The next problem shows in particular that any smooth map from an m-manifold to a $2m$-manifold can be approximated by an immersion. Note that when a smooth map $F : \Lambda \times M \to N$ is regarded as a family of maps $F_\lambda : M \to N$ then it gives a smooth map $\Lambda \times TM \to TN$ by $(\lambda, v) \mapsto (DF_\lambda)(v)$.

17. Suppose that $F_0 : M \to N$ is a smooth map from a compact m-dimensional manifold to an n-dimensional manifold. Show that for every $a \in M$ the map F_0 belongs to some smooth family of maps F_λ such that the associated map $\Lambda \times (TU - 0_U) \to TN$ is transverse to 0_N for some neighborhood U of of a. Conclude from this and a compactness argument that F_0 belongs to some smooth family such that $\Lambda \times (TM - 0_M) \to TN$ is transverse to 0_N. It follows that for almost all $\lambda \in \Lambda$ (i.e. for all λ outside a set of measure zero) the map F_λ has no bad non-immersion points. Conclude that in the special case $n \geq 2m$ F_λ is an immersion for almost all λ.

18. Let F_0 be as in Problem 17. Show that F_0 belongs to some smooth family such that $\Lambda \times (TM - 0_M) \to TN$ is transverse to 0_N and $\Lambda \times (M \times M - \Delta_M) \to N \times N$ is transverse to Δ_N.