Spring 2015 MA 2110 Manifolds
Supplement # 5
Existence of Partitions of Unity

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1 Bump Functions

Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = e^{-1/x}$ if $x > 0$ and $f(x) = 0$ if $x \leq 0$. Let us see that this is smooth. The only question is about the behavior at $x = 0$. Note that where $x > 0$ the function is smooth and its nth derivative has the form $f^{(n)}(x) = R_n(x)e^{-1/x}$ with R_n a rational function. To show by induction that the nth derivative at 0 exists and is zero, it suffices to show that the right-hand derivative of $f^{(n-1)}$ at 0 exists and is zero. But this is the limit of $R_{n-1}(x)e^{-1/x}$.

The *support* of a continuous function $f : X \to \mathbb{R}$ is the closure of the set of all points $x \in X$ such that $f(x) \neq 0$. In other words a point belongs to the support of f if there is no neighborhood in which the function is identically zero. Denote it by *supp* f.

Using g above we can make a smooth nonnegative function h on $\mathbb{R}^n$, positive at 0, and supported in any given neighborhood of 0, by writing $h(x) = g(r^2 - |x|^2)$ for some small $r > 0$.

Now we can do the same thing in a smooth n-manifold M. For any point $p \in M$ and open neighborhood U of p, we make a smooth function $f : M \to \mathbb{R}$ such that $f \geq 0$, $f(p) > 0$, and *supp* $f \subset U$. To do so, first note that without loss of generality U is the domain of a smooth chart $x : U \to \mathbb{R}^n$ and $x(p) = 0$. Now let h be as above, supported in the neighborhood $x(U)$ of 0. Define $f(q) = h(x(q))$ if $q \in U$ and $f(q) = 0$ if $q \notin U$.

2 Definitions

A collection (S_α) of subsets of the space X is called *locally finite* if every point in X has a neighborhood N such that there are only finitely many α for which $N \cap S_\alpha \neq \emptyset$.

Recall that a space X is called *locally compact* if every point has arbitrarily small compact neighborhoods: for every $x \in X$, for every open set $U \subset X$ such that $x \in U$, there exist an open set V and a compact set K such that $x \in V \subset K \subset U$. (Some authors use a weaker condition: a space is locally compact if and only if every point has a compact neighborhood. For Hausdorff spaces the two definitions are equivalent.) Of course, manifolds are locally compact because $\mathbb{R}^n$ is.

A collection (f_i) of continuous functions on X is called *locally finite* if the collection of sets (*supp* f_i) is locally finite. If this is the case, then the (pointwise) sum $\Sigma_i f_i$ is well-defined and continuous, since every point has a neighborhood in which the sum is really a finite sum.

A continuous *partition of unity* on a space X is a locally finite collection of continuous real-valued functions $f_i \geq 0$ such that $\Sigma_i f_i = 1$. If X is a smooth manifold and the functions are smooth then the partition of unity is said to be smooth.

A partition of unity (f_i) is *subordinate to* an open cover $\mathcal{O}$ if for every i the support of f_i is contained in some element of $\mathcal{O}$.

3 Existence of partitions of unity

Theorem 1 *If $\mathcal{O}$ is any open cover of the smooth manifold X, then X has a smooth partition of unity subordinate to $\mathcal{O}$.*

The proof uses the following lemma.

Lemma 1 *If the space X is locally compact, second countable, and Hausdorff, then there exist a sequence (K_j) of compact sets in X and a sequence (Ω_j) of open sets in X such that $K_j \subset \Omega_j$, and such that every point of X belongs to K_j for at least one j, and such that the collection (Ω_j) is locally finite.*

Proof of Lemma: By assumption the topology of X has a countable basis. That is, there is a countable set $\mathcal{B}$ of open sets such that for every neighborhood N of every point x in X there exists $U \in \mathcal{B}$ with $x \in U \subset N$. Because every point has a compact neighborhood, it follows that every point belongs to a precompact element of $\mathcal{B}$. (A subset of a topological space is *precompact* if its closure is compact.) Thus X is covered by some infinite sequence $U_1, U_2, \ldots$ of precompact open sets.

Let V_i be $U_1 \cup \ldots \cup U_i$. This gives a nested sequence of precompact open sets $V_1 \subset V_2 \subset \ldots$ which again covers X.

Next we find a sequence $W_1 \subset W_2 \subset \ldots$ of open subsets covering X such that $\bar{W}_j \subset W_{j+1}$ for all j. This can be constructed as a subsequence (V_{i_j}), because for each i there is some $i' > i$ such that the compact set $\bar{V}_i$ is contained in $V_{i'}$.

For convenience, extend the sequence to the left by writing $W_j = \emptyset$ for $j \leq 0$.

Now define $K_j = \bar{W}_j - W_{j-1}$ and $\Omega_j = W_{j+1} - \bar{W}_{j-2}$. Note that each point belongs to at most three of the sets Ω_j; in fact, if $k - j \geq 3$ then $\Omega_j \cap \Omega_k \subset W_{j+1} \cap (X - \bar{W}_{k-2}) = \emptyset$.

QED

Proof of Theorem: Let K_j and Ω_j be as in the Lemma.

First fix j. For every $x \in K_j$ there is a smooth function $f_{x,j} \geq 0$ on X such that $f_{x,j}(x) > 0$ and such that the support of $f_{x,j}$ is contained in Ω_j and also contained in some element of $\mathcal{O}$. Because K_j is compact, we may choose a finite set F_j of points in K_j in such a way that the sum of the functions $f_{x,j}$ over $x \in F_j$ is positive at every point of K_j.

Now let j vary. Consider the functions $f_{x,j}$, where j is arbitrary and $x \in F_j$. They form a locally finite family, so we may add them up and get a continuous function f. This is positive at every point of X. The functions $f_{x,j}/f$ form a smooth partition of unity. Each $f_{x,j}/f$ has its support in some element of $\mathcal{O}$, so the partition of unity is subordinate to $\mathcal{O}$.

QED

Remark: A similar argument shows that if X is any locally compact second countable Hausdorff space then there is a continuous partition of unity subordinate to any given open cover. This requires being able to make continuous functions to play the role of the functions $f_{x,j}$ in the proof above. For any $x \in X$ there is a continuous $f \geq 0$ that is positive at x and is supported in any given neighborhood of x, by the Tietze Extension Theorem.

Remark: The Lemma above also implies that every locally compact second countable Hausdorff space is *paracompact*: every open cover has a locally finite refinement. (An open cover $\mathcal{O}'$ is a *refinement* of $\mathcal{O}$ if for every $U \in \mathcal{O}'$ there exists $V \in \mathcal{O}$ such that $U \subset V$.)