Spring 2015
MA 2110, Introduction to Manifolds
Supplement #2
Manifolds in $\mathbb{R}^m$ and applications of the Inverse Function Theorem

January 29, 2015

Here is a discussion of the concept of m-dimensional smooth submanifold of $\mathbb{R}^n$. It will be superseded by the general concept of submanifold of an abstract manifold, but right now I want to get some ideas across by looking at this concrete case.

Apart from the Chain Rule, the main tool here is the Inverse Function Theorem.

1 Curves in the plane

Let us start with the case $m = 1$, $n = 2$. I will give several definitions of *smooth curve in the plane* and show that they are all logically equivalent.

Let $C \subset \mathbb{R}^2$ be a subset and let $p = (a, b) \in C$ be a point. The following four conditions are equivalent, and if they hold for every $p \in C$ we say that C is a smooth curve in $\mathbb{R}^2$.

(1) (Locally C is related to a line by a diffeomorphism of the ambient space.) For some open set $U \subset \mathbb{R}^2$ containing p there exists a diffeomorphism $\Phi : U \to \Phi(U)$ from U to some open subset of $\mathbb{R}^2$ such that $\Phi(C \cap U) = (\mathbb{R} \times 0) \cap \Phi(U)$ and $\Phi(p) = (0, 0)$.

(2) (Locally C can be given a regular parametrization.) For some open subset V of $\mathbb{R}$ there is a smooth map $\phi : V \to \mathbb{R}^2$ such that $\phi(V)$ is a neighborhood of p in C, $\phi(0) = p$, and the derivative (or "velocity vector") $\phi'0) \in \mathbb{R}^2$ is not zero.

(3) (Locally C is a graph.) For some open set $U \subset \mathbb{R}^2$ containing p, either the set $U \cap C$ can be described as the set of all pairs (x, y) with $y = f(x)$ and $x \in J$ for some smooth f defined in an open interval $J \in \mathbb{R}$, or it can be described as the set of all pairs (x, y) with $x = g(y)$ and $y \in J$ for some open interval $J \in \mathbb{R}$.

(4) (Locally C is a regular level set.) For some open set $U \subset \mathbb{R}^2$ containing p there exists a smooth map $\psi : U \to \mathbb{R}$ such that $\psi^{-1}(0) = U \cap C$ and $D_p\psi$ is not zero.

We outline the proof:

(1) implies (2) directly. In fact, define ϕ by $\phi(u) = \Phi^{-1}(u, 0)$ (the domain V being the set of all $u \in \mathbb{R}$ such that $(u, 0) \in \Phi(U)$).

(2) implies (3) using the Inverse Function Theorem in one dimension. To see this, write $\phi(u) = (\phi_1(u), \phi_2(u))$. Either $\phi_1'(0)$ or $\phi_2'(0)$ is nonzero, say the former. Restricting to a smaller interval if necessary, we can assume that ϕ_1 has an inverse. Let f be $\phi_2 \circ \phi_1^{-1}$. Or if $\phi_2'(0) \neq 0$ then locally ϕ_2 has an inverse and we let g be $\phi_1 \circ \phi_2^{-1}$.

(3) easily implies (4). In fact, if C is described locally by $y = f(x)$ then let $\psi(x, y) = y - f(x)$. If it is described by $x = g(y)$ then let $\psi(x, y) = x - g(y)$.

(4) implies (1) using the Inverse Function Theorem in two dimensions. In fact, one of the partial derivatives of ψ at P is nonzero, say $\frac{\partial \psi}{\partial y}$. Define $\Phi : U \to \mathbb{R}^2$ by $\Phi(x, y) = (x - a, \psi(x, y))$. The derivative of Φ at P is an invertible two by two matrix, so after restricting to a smaller open neighborhood of a the map Φ becomes a diffeomorphism onto its image.

2 The general case

Now let $0 \leq m \leq n$. Let $M \subset \mathbb{R}^n$ be a subset and let $p \in M$ be a point. The following four conditions are equivalent, and if they hold for every $p \in M$ we say that M is a smooth m-dimensional manifold in $\mathbb{R}^n$.

(1) For some open set $U \subset \mathbb{R}^n$ containing p there exists a diffeomorphism $\Phi : U \to \Phi(U)$ from U to some open subset of $\mathbb{R}^n$ such that $\Phi(M \cap U) = (\mathbb{R}^m \times 0) \cap \Phi(U)$ and $\Phi(p) = 0$.

(2) (Regular parametrization) For some open neighborhood V of 0 in $\mathbb{R}^m$ there is a smooth map $\phi : V \to \mathbb{R}^n$ such that $\phi(V)$ is a neighborhood of p in M, $\phi(0) = p$, and the derivative $D_0\phi$ (or equivalently the corresponding $n \times m$ matrix of partial derivatives) has rank m (the maximum possible).

(3) (Graph) For some open set $U \subset \mathbb{R}^n$ containing p, the set $U \cap M$ is related by some permutation of the n standard coordinates in $\mathbb{R}^n$ to the set of all pairs $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ with $y = f(x)$, for some smooth map $f : W \mapsto \mathbb{R}^{n-m}$ whose domain W is an open set in $\mathbb{R}^m$.

(4) (Regular level set) For some open set $U \subset \mathbb{R}^n$ containing p there exists a smooth map $\psi : U \to \mathbb{R}^{n-m}$ such that $\psi^{-1}(0) = U \cap M$ and such that the derivative $D_p\psi$ (or equivalently the corresponding $(n - m) \times n$ matrix of partial derivatives) has rank $n - m$ (the maximum possible).

The arguments are essentially the same as in the case $m = 1$, $n = 2$. Here are details for the two most interesting steps.

(2) implies (3) using the Inverse Function Theorem in m dimensions. After some permutation of coordinates we can assume that the first m rows of the matrix of $D_0\phi$ constitute an invertible $m \times m$ matrix. Write $\phi(u) = (\phi_1(u), \phi_2(u)) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, so that $D_0\phi_1$ is invertible. Restricting to a smaller domain if necessary, we can assume that ϕ_1 has an inverse. Let f be $\phi_2 \circ \phi_1^{-1}$.

(4) implies (1) using the Inverse Function Theorem in n dimensions. After composing with a permutation we can assume that the last $n - m$ columns of the matrix for $D_p\psi$ constitute an invertible $(n-m) \times (n-m)$ matrix. Define $\Phi : U \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ by $\Phi(x, y) = (x - a, \psi(x, y))$ where $p = (a, b)$. The derivative of Φ at p is an invertible $n \times n$ matrix, so after restricting to a smaller open neighborhood Φ becomes a diffeomorphism to its image.

Remark: "(4) implies (3)" is a version of the Implicit Function Theorem.

3 Tangent Spaces

To a smooth m-manifold $M \subset \mathbb{R}^n$ and a point $p \in M$ is associated an m-dimensional vector subspace of $\mathbb{R}^n$, the *tangent space* T_pM. Let us describe it using (1) above.

If Φ is a diffeomorphism as in (1) then we let T_pM be $(D_p\Phi)^{-1}(\mathbb{R}^m \times 0) = (D_0\Phi^{-1})(\mathbb{R}^m \times 0)$. To see that this is well-defined, first note that it does not change (because $D_p\Phi$ does not change) if Φ is replaced by its restriction to a smaller open neighborhood of P. Then suppose that Φ_1 and Φ_2 are two diffeomorphims as in (1) both having the same domain. Writing $\Phi_2 = h \circ \Phi_1$, thus $\Phi_1^{-1} = \Phi_2^{-1} \circ h$, we have $D_0\Phi_1^{-1} = D_0\Phi_2^{-1} \circ D_0h$. Since the diffeomorphism $h : \Phi_1(U) \to \Phi_2(U)$ preserves (a neighborhood of 0 in) $\mathbb{R}^m \times 0$, the linear isomorphism $D_0h : \mathbb{R}^n \to \mathbb{R}^n$ preserves $\mathbb{R}^m \times 0$ and therefore

$$(D_0\Phi_1^{-1})(\mathbb{R}^m \times 0) = (D_0\Phi_2^{-1})((D_0h)(\mathbb{R}^m \times 0)) = (D_0\Phi_2^{-1})(\mathbb{R}^m \times 0).$$

There are similar descriptions using (2), (3) or (4). We leave it as an exercise to show that given a map ϕ as in (2), T_pM is the image of $D_0\phi : \mathbb{R}^m \to \mathbb{R}^n$, and that given a map ψ as in (4) T_PM is the kernel of $D_p\psi : \mathbb{R}^n \to \mathbb{R}^{n-m}$.