SPRING 2015 MA 2110 MANIFOLDS
SUPPLEMENT #3
TOPOLOGY REVIEW

This is a summary of material, most of which you should have seen before. It is written very tersely and not intended as an introduction, but more as a resource for reminding yourself of things or filling in little gaps in your knowledge.

1. Spaces

A topological space consists of a set X, whose elements are called the points of the space, and a set of subsets of X which are called the open sets. There are three axioms:

1. The union of any set of open sets is open
2. The intersection of two open sets is open
3. X itself is open

Since the union of the empty set of sets is empty, Axiom 1 implies that the empty set is open. If we observe the convention that the intersection of the empty set of subsets of X is X, then Axioms 2 and 3 can be combined in a single statement:

2+3. The intersection of any finite set of open sets is open.

The set of all open sets is sometimes called the topology of the space. Thus a space consists of a set and a topology for that set. In practice one often uses the same name for the point set and for the space.

If $\mathcal{B}$ is a set of open sets of X, then $\mathcal{B}$ is called a *basis* for the topology if every open set is the union of some set of elements of $\mathcal{B}$. Of course, the topology is determined by the basis.

Even if a set $\mathcal{B}$ of subsets of X is not a basis for any topology, there is still a smallest topology for which the elements of $\mathcal{B}$ are open. The open sets are those subsets of X which may be expressed as the union of sets each of which is the intersection of finitely many sets belonging to $\mathcal{B}$. (One can check that this really is always a topology.) $\mathcal{B}$ is then called a subbasis for that topology. It is a basis if the intersection of two elements of $\mathcal{B}$ is always a union of elements of $\mathcal{B}$ and if X itself is a union of elements of $\mathcal{B}$.

Some topologies are determined by metrics. A metric on the set X is a function assigning to each pair of points a real number $d(x, y) \geq 0$ (which we call the distance between the points), subject to the following axioms:

$d(y, x) = d(x, y)$,
$d(x, z) \leq d(x, y) + d(y, z)$, $d(x, x) = 0$, and
$d(x, y) > 0$ if $x \neq y$.

A set $S \subset X$ is then called open if for every $x \in S$ there is some $r > 0$ such that S contains the ball $B_r(x) = \{y \in X : d(x, y) < r\}$. The balls are open, and the set of all balls is a basis.

The set $\mathbb{R}$ of real numbers, with the usual metric topology, is a key example. The open intervals constitute a basis.

A subset $N \subset X$ of a space is called a *neighborhood* of a point $x \in X$ if N contains an open set that contains x. Thus x belongs to every neighborhood of x. A set $S \subset X$ is open if and only if S is a neighborhood of every point of S. The set of all points $x \in X$ such that S is a neighborhood of x is called the *interior* of S. It is the union of all the open sets contained in S. Thus we can say that the interior of S is the largest open subset of X that is contained in S.

Given a subset $S \subset X$, a point $x \in X$ is called a *limit point* of S if every neighborhood of x has nonempty intersection with $S - \{x\}$. S is called *closed* if it contains all of its limit points. The *closure* of S is the union of S with the set of limit points of S. In other words, x belongs to the closure of S if and only if every neighborhood of x has nonempty intersection with S.

In other words, x is in the closure of S if and only if x has no neighborhood contained in the complement $X - S$. Thus S is closed if and only if every point in $X - S$ has a neighborhood inside $X - S$, if and only if $X - S$ is open. To sum up, S is closed if and only if $X - S$ is open.

The closure of S is the complement of the interior of $X - S$. It is the smallest closed set containing S, and it is the intersection of all the closed sets containing S.

(There is in general no smallest open set containing S, and no largest closed set contained in S.)

We sometimes have to consider two topologies on the same point set, and it may happen that every open set of the first topology is also open for the second. We say that one topology is larger (or finer) and the other is smaller (or coarser). (Probably it is not a good idea to say, as some do, that one topology is stronger or weaker than another, because historically these terms have not been used consistently. Presumably, those people who say that a finer topology is stronger are seeing the topology as something that keeps the points apart, while those who say that a coarser topology is stronger are seeing the topology as something that holds the points together.)

A given set X has a largest topology, namely the one in which all sets are open. This is called the discrete topology, and X is then called a discrete space. At the other extreme, the indiscrete topology (or trivial topology) has no open sets other than X and $\emptyset$.

2. Continuous maps

If X and Y are (the point sets of) two spaces, then a map $f : X \to Y$ is *continuous* if for every open set $U \subset Y$ the preimage $f^{-1}(U)$ is an open set of X. An equivalent condition is that for every closed set $C \subset Y$ the preimage $f^{-1}(C)$ is a closed set of X. Another equivalent condition is that for every subset $S \subset X$ the image of the closure of S is contained in the closure of the image of S.

We say that f is *continuous at* $x \in X$ if, for every neighborhood N of $f(x)$ in Y, the preimage $f^{-1}(N)$ is a neighborhood of x in X. Continuity of f is equivalent to continuity of f at every point in its domain. For metric spaces, continuity of f at a point a is equivalent to the usual $\epsilon - \delta$ condition: for every $\epsilon > 0$ there exists $\delta > 0$ such that if $d(x, a) < 0$ then $d(f(x), f(a)) < \epsilon$.

The composition of two continuous maps is always continuous. The identity map $1 : X \to X$ is always continuous. Thus we have a category whose objects are spaces and in which a morphism is a continuous map.

A homeomorphism between spaces X and Y is an isomorphism in this category, in other words a continuous map $f : X \to Y$ such that f has an inverse and this inverse is also continuous. In other words a homeomorphism is a bijection $f : X \to Y$ such that a set U is open in X if and only if $f(U)$ is open in Y.

3. Subspaces

If X is a space then any subset $A \subset X$ becomes a space by giving A the smallest topology such that the inclusion map $A \to X$ is continuous: call a subset S of A open if and only if it is the intersection of A with some open subset of X. This is called the subspace topology, or the relative topology. "Subspace of X" always means a subset of the space X equipped with the topology determined in this way by the topology of X.

Of course, when both X and A are being considered as spaces then in speaking of a set $S \subset A$ we have to be clear as to what we mean when we say that S is open, or closed; a subset of A might be open in the space A but not open in the space X.

By a slight generalization, if $f : Y \to X$ is any map of sets then a topology on X determines a topology on Y, the smallest such that f is continuous, by calling a subset of Y open if and only if it is the preimage of some open subset of X. If f is injective, then this makes Y homeomorphic to the subspace $f(Y)$. If it is not injective, then Y will be an odd sort of space, in which some one-point sets are not closed.

Note that when Y inherits its topology from X in this way then, for any space Z and any map $g : Z \to Y$, the map g is continuous if and only if the composed map $f \circ g : Z \to X$ is continuous. To express this more informally in the most important case, a map from Z to a subspace of X is continuous if and only if it is continuous as a map to X.

4. Quotient spaces

Now again suppose that a topology on X is given, and suppose we have a map of sets $f : X \to Y$. This determines a topology on Y, the largest such that f is continuous, by calling a subset of Y open if and only if its preimage is open in X.

Here the most important case is that in which f is surjective. Up to homeomorphism this means that Y can be made by specifying an equivalence relation on the set X and letting the set Y be the set of equivalence classes, and calling a subset of Y open if and only if the union of that set of subsets of X is open in X. Y is then called a quotient space of X.

Note that when Y inherits its topology from X in this way then, for any space Z and any map $h : Y \to Z$, h is continuous if and only if the composed map $h \circ f : X \to Z$ is continuous. To express this more informally in the most important case, making a continuous map from a quotient space of X to Z is the same as making a continuous map from X to Z that yields a well-defined map of sets on the quotient.

Note that we have three very closely related ideas:

1. If $X \to Y$ is a surjective map and X has a topology then we make a topology on Y called the quotient topology.

2. If X is a space and $\sim$ is an equivalence relation on the point set X then we denote the set of equivalence relations by $X/\sim$ and give this set a topology. This is a special case of 1. Up to homeomorphism every example of 1 is of this kind; for any surjection $f : X \to Y$,

we have a homeomorphism between Y (with the quotient topology) and the quotient space $X/\sim$ where $\sim$ is the equivalence relation given by $x \sim x'$ iff $f(x) = f(x')$.

3. If X and Y are spaces then a continuous surjective map $f : X \to Y$ is called a quotient map if the open sets $B \subset Y$ are the only subsets of Y such that $f^{-1}(B)$ is open in X. This is the same situation as 1 except that the point of view is now different: in 1 we were using the topology on X and the map f to make a topology on Y, whereas here the topologies on both X and Y are given and we are naming a property of the map.

Not every continuous surjection $f : X \to Y$ is a quotient map. A continuous map $f : X \to Y$ is called closed if for every closed subset $A \subset X$ the set $f(A)$ is closed in Y. A closed surjection is always a quotient map, because for any $B \subset Y$ if $f^{-1}(B) \subset X$ is closed then $B = f(f^{-1}(B)) \subset Y$ is closed. (But not every quotient map is closed. For example a linear projection of $\mathbb{R}^2$ to $\mathbb{R}$ is not closed, but it is a quotient map because it is open: the image of an open set is open.)

5. Connectedness

Call a space X disconnected if it is the union of some pair of disjoint nonempty open sets (equivalently if it is the union of a pair of disjoint nonempty closed sets, equivalently if it has a proper nonempty set that is both open and closed). Call X connected if it is not disconnected and not empty. Call a subset of a space connected if, topologized as a subspace, it is connected.

A one-point space is connected.

If $f : X \to Y$ is continuous and surjective, and if X is connected, then Y is connected. (Proof: Suppose for contradiction that $Y = A \cup B$ with $A \cap B = \emptyset$ and both A and B open and nonempty. Then $X = f^{-1}(A) \cup f^{-1}(B)$ with $f^{-1}(A) \cap f^{-1}(B) = \emptyset$ and both $f^{-1}(A)$ and $f^{-1}(B)$ open and nonempty.)

The union of two connected sets in a space is connected if the intersection is nonempty. (Proof: Suppose that $X \cap Y$ has a point p in it and that X and Y are connected. If $X \cup Y$ is the union of disjoint sets A and B, both open in $X \cup Y$, then p belongs to one of these, say A. $A \cap X$ is open and closed in X and nonempty, therefore equal to X. Likewise $A \cap Y = Y$. Thus $A = X \cup Y$.)

In any space X, define a relation between points by calling two points related if there exists a connected subset of X containing both of them. This is an equivalence relation. The equivalence classes are themselves connected. (Proof: If C is such a class, and if $C = A \cup B$ with A and B disjoint, nonempty, and open in C, then choose points $a \in A$ and $b \in B$. There is a connected set K in X containing both a and b. K is contained in C. K is the union of disjoint nonempty open sets $A \cap K$ and $B \cap K$, contradiction.) The equivalence classes are called the components, or connected components, of X. Every connected subset of X is contained in a (unique) component, and the components are the largest connected subsets of X.

The closure of a connected set is connected. (Proof: Assume that the closure of S is the union of two disjoint nonempty closed sets A and B. If S is connected then one of the sets $A \cap S$ and $B \cap S$ must be S, say the former. Then A contains S. Since A is closed in the closure of S, it must equal the closure.)

Any closed interval in $\mathbb{R}$ is connected. (Proof: Suppose for contradiction that the interval $[a, b]$ is the union of disjoint nonempty closed sets A and B. One of them, say A, contains the point a. The set B has a greatest lower bound c. The point c belongs to B because B is closed. Also, clearly $a < c \leq b$. But all of the points between a and c belong to A. Since A is closed as well, c belongs to A. Contradiction.)

A *path* in X from a point p to a point q is a continuous map $\alpha : [0, 1] \to X$ such that $\alpha(0) = p$ and $\alpha(1) = q$.

A nonempty space is called *path-connected* if there is a path from p to q for every p and every q. Because $[0, 1]$ is connected, every path-connected space is connected. Define an equivalence relation on the point set of X by saying that $p \sim q$ if there is a path from p to q in X. The equivalence classes are called path components. They are the maximal path-connected subspaces of X.

Convex subsets of $\mathbb{R}^n$ are path-connected and therefore connected.

It is an easy consequence of the completeness of the real numbers that a convex subset of $\mathbb{R}$ must be either the whole line, or a half-line (left or right, open or closed), or a bounded interval (open, closed, or "half-open"), or a single point, or the empty set.

The nonempty convex subsets of $\mathbb{R}$ are the only connected sets in $\mathbb{R}$, because if $S \subset \mathbb{R}$ is a nonconvex set then there exist $a < b < c$ with $a \in S$, $b \notin S$, and $c \in S$, in which case $S = (S \cap (\infty, b)) \cup (S \cap (b, \infty)))$ is disconnected.

The space X is *locally connected* if every point in X has arbitrarily small connected neighborhoods in X, in other words if every neighborhood of any point contains a connected neighborhood of the same point. This implies that the components of an open subset $U \subset X$ are always open. (Proof: Let C be a component of U. For any $x \in C$ there is a connected neighborhood N of x contained in U. N must be contained in C. Therefore C is a neighborhood of x. Since C is a neighborhood of each of its points, it is open.) The converse also holds. (Proof: If components of open sets are open, then let U be any open set containing the point x. The component of U containing x is open, so it is a connected neighborhood contained in U.)

6. Product spaces

If X and Y are spaces then the product space $X \times Y$ is defined to be the cartesian product of the point sets, with the smallest topology such that for U open in X and V open in Y the sets $U \times Y$ and $X \times V$ are open. Thus these sets form a subbasis and the sets $U \times V = (U \times Y) \cap (Z \times V)$ form a basis.

The projection maps $X \times Y \to X$ and $X \times Y \to Y$ are continuous. The topology is minimal such that they are continuous. Making a continuous map $Z \to X \times Y$ is equivalent to making continuous maps $Z \to X$ and $Z \to Y$.

One can also make the product of more than two spaces, even infinitely many.

If X and Y are spaces and $A \subset X$ and $B \subset Y$ are subsets, then the set $A \times B$ appears to have two topologies: as a subspace of $X \times Y$ and as a product of two subspaces. These are in fact equal. We can see this as follows. Temporarily write $(A \times B)_s$ for the subspace of the product and $(A \times B)_p$ for the product of the subspaces. The identity map $(A \times B)_s \to (A \times B)_p$ is continuous because the projections $(A \times B)_s \to A$ and $(A \times B)_s \to B$ are continuous, because the composed maps $(A \times B)_s \to A \to X$ and $(A \times B)_s \to B \to Y$ are continuous, because they can be factored $(A \times B)_s \to X \times Y \to X$ and $(A \times B)_s \to X \times Y \to Y$ as continuous inclusion followed by continuous projection. The identity map $(A \times B)_p \to (A \times B)_s$ is continuous because the inclusion $(A \times B)_p \to X \times Y$ is continuous, because the composed maps $(A \times B)_p \to X$ and $(A \times B)_p \to Y$ are continuous, because they can be factored $(A \times B)_p \to A \to X$ and $(A \times B)_p \to B \to Y$ as continuous projection followed by continuous inclusion.

7. Hausdorff spaces

The space X is called a *Hausdorff space* if points can be separated by open sets, i.e. if for distinct points p and q in X there always exist disjoint open sets U and V in X such that $p \in U$ and $q \in V$. This is equivalent to saying that in the product space $X \times X$ the diagonal set $\{(x, x)\} : x \in X$ is closed.

Metric spaces are Hausdorff. Subspaces of Hausdorff spaces are Hausdorff. Products of Hausdorff spaces are Hausdorff.

8. Compactness

A space X is *compact* if every open cover has a finite subcover. This means that, whenever a collection $\mathcal{U}$ of open subsets of X is such that X is the union of all of the elements of $\mathcal{U}$, then $X = V_1 \cup \cdots \cup V_N$ for some $N \geq 0$ and some $V_1, \ldots, V_N$ belonging to $\mathcal{U}$. If $\mathcal{B}$ is a basis for X then X is compact if every open cover by elements of $\mathcal{B}$ has a finite subcover.

If A is a subset of the space X then compactness of the subspace A can be discussed without explicitly mentioned the subspace topology: A is compact if and only if whenever A is contained in the union of a collection of open subsets of X then it is contained in the union of a finite subcollection.

If $f : X \to Y$ is continuous and $A \subset X$ is compact, then $f(A) \subset Y$ is compact.

A closed subset of a compact space is always compact.

A compact subset of a Hausdorff space is always closed. Indeed, if K is compact in X and X is Hausdorff then for any point $p \in X - K$ there are disjoint open sets in X containing p and K. (Proof: For $k \in K$ there are disjoint open sets U_k and V_k in X such that $p \in U_k$ and $k \in V_k$. There is a finite collection $k_1, \ldots, k_N$ such that K is contained in the union of the V_{k_i}. The point p belongs to the intersection of the U_{k_i}, which is open, and which is disjoint from the union of the V_{k_i}, which contains K.)

A closed interval is compact. (Proof: Suppose that $[a, b]$ is covered by a collection $\mathcal{J}$ of open intervals. Let T be the set of all $x \in [a, b]$ such that the interval $[a, x]$ is covered by finitely many of those intervals. T is convex, and is a subset of $[a, b]$, and contains a. Thus it is either $[a, c]$ or $[a, c)$ for some c such that $a < c \leq b$. Choose $J \in \mathcal{J}$ such that $c \in J$. Choose $d \in J \cap T$ less than c. The set $[a, d]$ is covered by finitely many of the given intervals. If $c < b$ then those same intervals plus J cover some interval $[a, e]$with $c < e \leq b$, contradicting the choice of c. Thus $c = b$, and furthermore $b \in T$ since again those same intervals plus J cover $[a, b]$.)

The product of two compact spaces is compact. (Proof: Suppose that X and Y are compact and that $X \times Y$ is covered by some basic open sets $U_i \times V_i$. For each $x \in X$, Y is covered by those V_i for which $x \in U_i$. Choose a finite set I_x of such i, such that Y is covered by those V_i with $i \in I_x$. The intersection of the U_i for $i \in I_x$ is an open set U_x containing x. Some finite collection (U_{x_j}) of these open sets covers X, and this leads to a finite subcover of the original cover of $X \times Y$, namely the collection of sets $U_i \times V_i$ with $i \in I_{x_j}$ for some j.)

Remark: The product of infinitely many compact spaces is also compact, but the proof is harder.

If K is compact and $x \in X$ then any open set in $X \times K$ that contains $x \times K$ must contain $U \times K$ for some open set U containing the point x. (Proof: Cover $x \times K$ by products $U_i \times V_i$ that are contained in the given open set. Since K is compact, a finite collection of the V_i covers K. Choose U to be the intersection of the corresponding finitely many U_i.) This is sometimes called the *Tube Theorem*. An equivalent statement is that the projection $X \times K \to X$ is a closed map if K is compact.

$\mathbb{R}^n$ topologized in the usual metric fashion is the same as the product of n copies of the space $\mathbb{R}$. It follows that every closed and bounded subset of $\mathbb{R}^n$ is compact, because it is a closed subset of a rectangle $[-c, c]^n$ and the rectangle is a product of compact intervals. Conversely, every compact set in $\mathbb{R}^n$ is closed and bounded. This converse holds in any metric space, not just in $\mathbb{R}^n$.

A continuous map $f : X \to Y$ from a compact space X to a Hausdorff space Y is always a closed map. (Proof: $A \subset X$ closed implies A compact, which implies $f(A)$ compact, which implies $f(A) \subset Y$ closed.) In particular f is a quotient map if it is a surjection, and f is a homeomorphism if it is a bijection.

A space is called *locally compact* if every point $x \in X$ has arbitrarily small compact neighborhoods, that is, if whenever U is an open set of X containing x then there is a compact set $K \subset X$ such that K contains some open set containing x. This includes compact Hausdorff spaces and more generally open subsets of compact Hausdorff spaces.

9. Function spaces

If X is a space and K is a locally compact space then let X^K be the set of all continuous maps $f : K \to X$, considered as a space by taking the following sets $\Omega_{A,U}$ as a subbasis: for a compact set $A \subset K$ and an open set $U \subset X$ define $\Omega_{A,U}$ to be the set of all $f \in X^K$ such that $f(A) \subset U$. The evaluation map $K \times X^K \to X$ is continuous. (Proof: To prove continuity at the point (k, f), take any open $U \subset X$ containing $f(k)$ and choose a compact neighborhood N of k in K that is contained in the open set $f^{-1}(U)$. The evaluation map takes $N \times \Omega_{N,U}$ into U.) Therefore for any space Y a continuous map $F : Y \to X^K$ determines a continuous map $\tilde{F} : K \times Y \to X$, by composing the evaluation map with $1 \times F : K \times Y \to K \times X^K$, in other words by writing $\tilde{F}(k, y) = F(y)(k)$. Conversely every continuous map $G : K \times Y \to X$ is $\tilde{F}$ for some F as above. (Proof: Define $F(y)(k) = G(k, y)$. Certainly for each y the map $F(y) : K \to X$ is continuous. We must verify that F is a continuous map from Y to X^K. If $y \in F^{-1}(\Omega_{A,U})$ then $A \times y$ is contained in the open set $G^{-1}(U)$, so by the Tube Theorem there is an open set $V \subset Y$ such that $A \times V \subset G^{-1}(U)$, and therefore $V \subset F^{-1}(\Omega_{A,U})$. Thus $F^{-1}(\Omega_{A,U})$ is open.)

10. Quotients and products

If $q : X \to Y$ is a quotient map and K is a locally compact space, then $q \times 1 : X \times K \to Y \times K$ is also a quotient map. (Proof: Suppose we have a map $Y \times K \to Z$ such that the composed map $X \times K \to Y \times K \to Z$ is continuous. This corresponds to a map $Y \to Z^K$ such that the composed map $X \to Y \to Z^K$ is continuous. Since q is a quotient map, it follows that $Y \to Z^K$ is continuous. From that it follows that $Y \times K \to Z$ is continuous. This implication means that $X \times K \to Y \times K$ is a quotient map.)