Fall 2009 MA 271 Stable Homotopy: Suggested Homework Problems

September 23, 2009

Sept.10

1. For spaces X and Y, let X^Y be the set of all continuous maps $f : Y \to X$, equipped with the compact-open topology, the smallest topology such that for every compact subset $K \subset Y$ and every open subset $U \subset X$ the set $\mathcal{O}_{K,U} = \{f \in X^Y | f(K) \subset U\}$ is open. Prove that every continuous map $Z \times Y \to X$ corresponds to a continuous map $Z \to X^Y$. Assuming that Y is locally compact (every neighborhood of every point contains a compact neighborhood of that point), prove the converse statement: a map of sets $Z \times Y \to X$ must be continuous if it corresponds to a continuous map $Z \to X^Y$. (Note that for this it suffices to show that the evaluation map $X^Y \times Y \to X$ is continuous.)

Sept. 15

2. Using Problem 1, verify that for a locally compact space Y there is a functor $X \mapsto X^Y$ right adjoint to the functor $Z \mapsto Z \times Y$.

3. Find left adjoints for some 'forgetful' functors, for example, the functor from spaces to sets that forgets the topology, the functor from abelian groups to sets that forgets the addition law, the functor from pointed spaces to spaces that forgets the base point. Find a left adjoint for the "unit group"functor from associative unital rings to groups.

4. Observe that when $L : \mathcal{C} \to \mathcal{D}$ is left adjoint to $R : \mathcal{D} \to \mathcal{C}$ then as a special case of the correspondence between $Mor_{\mathcal{D}}(LX, Y)$ and $Mor_{\mathcal{C}}(X, RY)$ we get a canonical element of $Mor_{\mathcal{C}}(X, RLX)$ corresponding to the identity map $LX \to LX$. Show that this map $X \to RLX$, the unit of the adjoint functor pair, is a natural transformation from the identity $1_{\mathcal{C}}$ to $R \circ L$. Likewise one defines the co-unit from $L \circ R$ to $1_{\mathcal{D}}$. Show that the bijection $Mor_{\mathcal{D}}(LX, Y) \cong Mor_{\mathcal{C}}(X, RY)$ is determined by the unit and co-unit.

5. Think through the details of the change-of-basepoint map $\phi_\gamma : \pi_n(X, x_0) \to \pi_n(X, x_1)$ determined by a path γ in X from x_0 to x_1. In particular, see that (1) it is well-defined, (2) it is a homomorphism, (3) it depends only on the homotopy class (relative to endpoints) of γ and thus

can be written $\phi_{[\gamma]}$, (4) it preserves multiplication of paths, $\phi_{[\gamma][\delta]} = \phi_{[\delta]} \circ \phi_{[\gamma]}$, and (5) constant paths have trivial effect, $\phi_{[c]} = 1$. It follows that it is an isomorphism, and that in the case $x_0 = x_1$ it gives an action of the fundamental group on $\pi_n(X, x_0)$.

6. Show that when $X = \Omega Y$ then this action of $\pi_1(X) = \pi_2(Y)$ on $\pi_n(X) = \pi_{n+1}(Y)$ is trivial.

7. Show that the Hurewicz map $h_n : \pi_n(X, x_0) \to H_n(X, x_0)$ is a homomorphism for every $n \geq 1$.

Sept. 17

8. Complete the proof of the long exact homotopy sequence of a pair by showing that kernel equals image in

$$\pi_n(Y, y_0) \to \pi_n(X, y_0) \to \pi_n(X, Y, y_0)$$

9. Compare two definitions of a "suspension map"$\pi_n(X) \to \pi_{n+1}(S^1 \wedge X)$: One definition uses adjointness of loopspace and (reduced) suspension. The unit (Problem 4) of this adjunction gives a map

$$\pi_n(X) \to \pi_n(\Omega(S^1 \wedge X)) = \pi_{n+1}(S^1 \wedge X)$$

Another definition uses the expression of the (unreduced) suspension as union $\Sigma X = C_+ X \cup C_- X$ of two cones with intersection X:

$$\pi_n(X) \cong \pi_{n+1}(C_- X, X) \to \pi_{n+1}(\Sigma X, C_+ X) \cong \pi_{n+1}(\Sigma X)$$

Show that (after applying the canonical map $\Sigma X \to S^1 \wedge X$) this agrees with the first definition.

Sept. 22

10. Define $\pi_n(f, y)$, for $f : Y \to X$ and $y \in Y$, as π_{n-1} of the homotopy fiber of f with respect to $f(y)$ (basepoint given by y). Obtain a long exact sequence $\ldots \to \pi_n(Y, y) \to \pi_n(X, f(y)) \to \pi_n(F, y) \to \pi_{n-1}(Y, y) \to \ldots$.

11. For maps $g : Z \to Y$ and $f : Y \to X$, show that $f \circ g$ is k-connected if both f and g are k-connected. Show that f is k-connected if $f \circ g$ is k-connected and g is $(k-1)$-connected. Show that g is k-connected if $f \circ g$ is k-connected and f is $(k+1)$-connected.