THE CLASSIFICATION OF COVERING SPACES

Let $X$ be a space, let $x_0$ be a point of $X$, and let $G$ be the fundamental group $\pi_1(X, x_0)$. I will define a functor

$$F : Cov_X \to Set$$

where $Cov_X$ is the category of covering spaces $(E, p : E \to X)$ of $X$ (a full subcategory of the comma category $Top \downarrow X$ of spaces over $X$) and $SetG$ is the category of right $G$-sets.

We will frequently use this pair of lifting lemmas, which apply to all covering spaces $p : E \to X$.

Lemma 1 (path lifting) If $\alpha : I \to X$ is continuous and $e \in E$ is given, with $p(e) = \alpha(0)$, then there exists a unique $\tilde{\alpha} : I \to E$ such that $p \circ \tilde{\alpha} = \alpha$ and $\tilde{\alpha}(0) = e$.

Lemma 2 (homotopy lifting for paths) If $H : I \times I \to X$ is continuous and $e \in E$ is given, with $p(e) = H(0, 0)$, then there exists a unique $\tilde{H} : I \times I \to E$ such that $p \circ \tilde{H} = H$ and $\tilde{H}(0, 0) = e$.

For each point $x \in X$ the set $p^{-1}(x) \subset E$ is called the fiber of $p$ over $x$.

These lifting lemmas lead to the following construction: Given a path $\alpha$ from $x$ to $x'$ in $X$, we get a map of sets $\alpha_* : p^{-1}(x) \to p^{-1}(x')$ by letting $\alpha_*(e) = \hat{\alpha}(1)$ where $\hat{\alpha}$ is chosen so that $\hat{\alpha}(0) = e$. Call this map transport along the path $\alpha$. We see that $(\alpha \beta)_*$ is the composition of $\alpha_*$ and $\beta_*$ (actually in the other order, because of conventions about multiplying paths and composing mappings). So write the operation on the right: $\alpha_*$ takes $e$ to $ea_\alpha$. Then

$$(e\alpha)_* \beta_* = e(\alpha \beta)_*$$

The second lemma insures that $\alpha_*$ depends only on the homotopy class (with endpoints fixed) of $\alpha$, so that if $a = [\alpha]$ we can write $e \mapsto ea$ instead of $e \mapsto ea_\alpha$.

Here $\alpha$ was a path, not necessarily a loop. If we specialize to the case where $\alpha$ is a loop based at $x_0$, then $a$ is an element of $G = \pi_1(X, x_0)$ and the transport construction becomes a right action of $G$ on the fiber $p^{-1}(x_0)$.

Now we define the functor $F$.

On objects, let $F(E, p)$ be the $G$-set $p^{-1}(x_0)$ just described.

On morphisms, if $f : (E, p : E \to X) \to (E', p' : E' \to X)$ is a map of covering spaces, first observe that $f$ takes the set $p^{-1}(x_0)$ into the set $p'^{-1}(x_0)$ (because $p' \circ f = p$) and then observe that this map is a $G$-map: $f(ea) = f(e)a$. This is so because a representative loop $\alpha$ for $a$ has a lifting $\tilde{\alpha}$ to $E$ that starts at $e$ and ends at $ea$, whence it also has a lifting $f \circ \tilde{\alpha}$ to $E'$ that starts at $f(e)$ and ends at $f(ea)$. Let $F(f)$ be that $G$-map.

Clearly $F$ is a functor: it takes compositions $(E, p) \to (E', p') \to (E'', p'')$ to compositions $p^{-1}(x_0) \to p'^{-1}(x_0) \to p''^{-1}(x_0)$ and identity maps $(E, p) \to (E, p)$ to identity maps.

HYPOTHESIS 1: $X$ is path-connected.
With this hypothesis, we now show that the functor $F$ is faithful. Given two maps $f$ and $g$ from $(E,p)$ to $(E',p')$, and assuming that $F(f) = F(g)$, we must show that $f = g$. In other words, assuming that $f(e) = g(e)$ for every $e \in E$ such that $p(e) = x_0$, we must show that $f(e) = g(e)$ for every $e \in E$.

Take any $e \in E$. Choose a path $\alpha$ in $X$ from $p(e)$ to $x_0$. Let $\tilde{\alpha}$ be the lifting to $E$ with $\tilde{\alpha}(0) = e$. The reverses of both $f \circ \tilde{\alpha}$ and $g \circ \tilde{\alpha}$ are liftings of the reverse of $\alpha$ to $E'$, and they both start at $f(\tilde{\alpha}(1)) = g(\tilde{\alpha}(1))$. Therefore (by the uniqueness in Lemma 1) they are equal. They end at $f(e)$ and $g(e)$ respectively, so $f(e) = g(e)$.

We can express this argument a little differently, more formally, by using the observation that the equation $f(e)a = f(ea)$ is valid for path classes, not just loop classes. Thus the value of $f$ at any $e \in E$ is determined by the values of $f$ in the fiber over $x_0$ because, choosing a path class $a$ from $p(e)$ to $x_0$, we have $f(e) = f(ea)a^{-1}$ where $ea \in p^{-1}(x_0)$.

**HYPOTHESIS 2:** $X$ is locally path-connected.

(By definition, this means that every point in $X$ has arbitrarily small path-connected neighborhoods.)

With this hypothesis we will prove that $F$ is fully faithful. Again let $(E,p)$ and $(E',p')$ be covering spaces of $X$. We have to show that every $G$-map $f_0$ from $p^{-1}(x_0)$ to $p'^{-1}(x_0)$ is $F(f)$ for some map $f : E \to E'$ of covering spaces.

We take our cue from the proof of faithfulness. Define a map of sets $f : E \to E'$ by $f(e) = (f_0(ea))a^{-1}$, where $a$ is any path class from $p(e)$ to $x_0$ in $X$. This is well-defined because $f_0$ has been assumed to be a $G$-map: If $b$ is a different choice of path class from $p(e)$ to $x_0$ in $X$, then $c = b^{-1}a$ is an element of $G$ and we have $a = bc$,

$$f_0(ea)a^{-1} = f_0(ebc)a^{-1} = f_0(eb)ca^{-1} = f_0(eb)b^{-1}$$

Certainly $p'(f(e)) = p(e)$. Certainly $f(e) = f_0(p)$ when the latter is defined, because then $a$ can be chosen to be the class of the constant path at $x_0$. Thus, if we can show that $f$ is continuous, it will be a morphism of $\text{Cover}_X$ such that $F(f) = f_0$ as desired.

To prove the continuity, we use the local path-connectedness of $X$.

Note that $f$ is compatible with transport along all paths in $X$: If $b$ is the class of a path from $x_1$ to $x_2$ then, choosing any $a_1$ from $x_1$ to $x_0$ and defining $a_2$ from $x_2$ to $x_0$ by $a_2 = b^{-1}a_1$, we have

$$f(eb) = f_0(eba_2)a_2^{-1} = f_0(ea_1)a_2^{-1} = f(e)a_1a_2^{-1} = f(e)b$$

To prove the continuity at a given $e \in E$, choose a neighborhood $N$ of $p(e)$ in $X$ such that it is evenly covered by both $p$ and $p'$. Then $p^{-1}(N)$ is homeomorphic, by a homeomorphism that respects the projections to $N$, with the product $N \times S$ of $N$ with a discrete space $S$, and similarly $p'^{-1}(N)$ is $N \times S'$. The statement to be proved, that $f$ is continuous at $e$, means that at a certain point in $N \times S$ a certain map $N \times S \to N \times S'$ is continuous. Write this map as $(x,s) \mapsto (a(x,s),b(x,s))$. 

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Clearly \( a(x, s) = x \), so that the map \( a \) is continuous. Since any path in \( N \times S \) is constant in the \( S \) factor, the compatibility with transport means that for every \( s \in S \) the map \( x \mapsto b(x, s) \) is constant on path-components of \( N \), therefore locally constant (this is where the local path-connectedness is used), therefore continuous, so that the map \( b \) is continuous.

At this point we have that \( F \) is fully faithful (under these two hypotheses on \( X \)). One consequence is that two covering spaces of \( X \) are isomorphic if and only if the give rise to isomorphic \( G \)-sets.

The most important covering spaces \((E, p)\) are those for which \( E \) is path-connected. Observe that these are precisely the ones for which the \( G \)-set \( F(E, p) \) is transitive. Indeed, transitivity is equivalent to saying that any two points in the fiber \( p^{-1}(x_0) \) are related by a path in \( E \); and since every path in \( E \) is related to some point in that fiber (by the path-connectedness of \( X \)) this in turn is equivalent to saying that any two points in \( E \) are joined by a path.

(By the way, since \( X \) is assumed locally path-connected it follows that \( E \) is locally path-connected; and this implies that \( E \) is path-connected if it is connected.)

Recalling the relationship between transitive \( G \)-sets and subgroups of \( G \), we now have a one-to-one correspondence between connected covering spaces of \( X \) (up to isomorphism) and at least some subgroups of \( G \) (up to conjugacy). (We have not yet addressed the question of whether every subgroup corresponds to a covering space.) Denote by \( E_H \) the covering space corresponding to the subgroup \( H \), if it exists.

We now work out what \( H \) is in terms of \( E_H \). It is the stabilizer group of the action of \( G \) on \( p^{-1}(x_0) \). It depends on a choice of point \( e_0 \) in that fiber. The loops in \( X \) whose classes fix \( e_0 \) are those which lift to loops based at \( e_0 \), in other words those whose classes belong to the subgroup \( p_\ast \pi_1(E_H, e_0) \) of \( G \). So that’s \( H \). Since the homomorphism \( p_\ast \) is injective (by Lemma 2), \( H \) is in fact isomorphic to \( \pi_1(E_H, e_0) \). Its index in \( G \) is the “number of sheets”, that is, the cardinality of the fiber.

Another consequence of the faithfulness is that the group of automorphisms of the covering space \( E_H \) is the group of automorphisms of the \( G \)-set \( H \setminus G \). This can be described as \( N_H G / H \), where \( N_H G \) is the normalizer of \( H \) in \( G \). Let’s work this out.

A \( G \)-map \( f \) from \( H \setminus G \) to a \( G \)-set \( S \) determines an element \( s = f(H) \in S \). This element is fixed by \( h \in H \):

\[
sh = f(H)h = f(Hh) = f(H) = s.
\]

The map \( f \) is in turn determined by \( s \), because \( f(Hg) = f(H)g = sg \) for all \( g \in G \). Note that, as a definition of a map \( f \), the equation \( f(Hg) = sg \) is unambiguous only if \( s \) is fixed by \( H \): We need that \( sg = sg’ \) whenever \( Hg = Hg’ \), that is, we need that \( sg = shg \) for every \( g \in G \) and \( h \in H \), that is, we need that \( s = sh \) for every \( h \in H \). Moreover, given any \( s \) fixed by \( H \), the map of sets defined by \( f(Hg) = sg \) will be a \( G \)-map taking \( H \) to \( s \). So the \( G \)-maps \( H \setminus G \) correspond bijectively with the elements of \( S \) that are fixed by \( H \).

In the special case \( S = K \setminus G \) the elements fixed by \( H \) are the cosets \( Ka \) such that for every \( h \in H \) we have \( Kah = Ka \), that is, \( aHa^{-1} \subset K \). Thus a map of covering spaces \( E_H \to E_K \) corresponds to a \( G \)-map \( H \setminus G \to K \setminus G \), and such a map exists if and only if some conjugate of \( H \) is contained in \( K \).
Let’s look closely at the special case $H = K$. A $G$-map $H \setminus G \to H \setminus G$ has the form $f_a : Hg \mapsto Hag$, where $a$ is such that $aHa^{-1} \subseteq H$. $f_a$ depends only on the coset $Ha$. The composition of two such maps is given by $f_a \circ f_b = f_{ab}$, because

$$f_a(f_b(Hg)) = f_a(Hbg) = Ha(bg) = H(ab)g = f_{ab}(Hg).$$

$f_a$ is always surjective. For it to be injective means that if $Ha = Hag$ then $H = Hg$, in other words if $g \in a^{-1}Ha$ then $g \in H$. Thus $aHa^{-1}$ must be equal to $H$, not just contained in it.

The elements $a \in G$ such that $aHa^{-1} = H$ form a subgroup of $G$ called the normalizer of $H$ and denoted $N_GH$. $H$ is normal in its normalizer. We find that the group of automorphisms of the $G$-set $H \setminus G$ is isomorphic to the quotient $N_GH/H$.

The group of automorphisms of a covering space is also called the group of covering transformations or deck transformations. Denote it by $\text{Deck}(E)$. So for a connected covering space we have $\text{Deck}(E_H) \simeq N_GH/H$.

A connected covering space $E_H$ is called regular if the corresponding subgroup $H \subseteq G$ is normal, or equivalently if the action of $\text{Deck}(E)$ on a fiber is transitive.

If $K/H \subseteq N_GH/H = \text{Deck}(E_H)$ is a subgroup of the group of automorphisms of a connected covering space of $X$, then the quotient space (orbit space) for this action of $K/H$ on $E_H$ is again a connected covering space of $X$. The fiber of the new covering space, as a (right) $G$-set, can be read off from the fiber of the old one: divide out $H \setminus G$ by the (left) action of $K/H$. This yields $K\setminus G$, so the new covering space is $E_K$.

As a special case, if $U$ is a simply-connected (universal) covering space of $X$ then $\text{Deck}(U) = G$ and the orbit space $H \setminus E$ for any subgroup is the covering space $E_H$.

We have not yet shown that a universal covering space exists, but, by the discussion above, if it does then every $E_H$ can be constructed from it as an orbit space. Thus the functor $F$ will be an equivalence of categories if $X$ has a universal covering space. To prove that it does, we need another assumption.

**HYPOTHESIS 3:** $X$ is semi-locally simply-connected.

This means that every point $x$ has a neighborhood such that every loop in that neighborhood becomes trivial in $\pi_1(X,x)$.

Let’s show that if $X$ satisfies all of these hypotheses then it has a simply-connected covering space $E$. To decide what the points in $E$ should be, we can work backwards: As soon as a point $e_0 \in E$ over $x_0 \in X$ is chosen, then the points of $E$ are going to have to correspond bijectively with the homotopy classes of paths in $X$ originating at $x_0$. That is because by Lemmas 1 and 2 the path classes originating at $x_0$ in $X$ are in bijection with the path classes in $E$ originating at $e_0$, while in the simply-connected space $E$ the path classes originating at $e_0$ are in bijection with the points. So we can define the set $E$ to be the set of all path classes $[\alpha]$ in $X$ with $\alpha(0) = e_0$. The problem is to find the right topology on this set. In any case it has a map $p : E \to X$ given by $p([\alpha]) = \alpha(1)$. 
Consider the following open sets \( U \subset X \): those such that \( U \) is path-connected and every loop in \( U \) is nullhomotopic in \( X \). Just for this proof these will be called the \textit{good} sets. By assumption, they form a basis for \( X \). If \( U \) is good, and if \( a = [\alpha] \) is a homotopy class of paths in \( X \) from \( x_0 \) to some point \( x \in U \), then let \( U_a \subset E \) be the set of all classes \( ab \) where \( b \) is represented by a path in \( U \). The map \( p \) takes \( U_a \) into \( U \), of course, and in fact this map \( U_a \to U \) is a bijection. It is surjective because \( U \) is path-connected. It is injective because if \( p(a[\beta_1]) = p(a[\beta_2]) \) for two paths \( \beta_1 \) and \( \beta_2 \) in \( U \) (both starting at \( x \)) then, since the loop \( \beta_1 \beta_2 \) is nullhomotopic, \( [\beta_1] = [\beta_2] \). We will use these sets \( U_a \) as basis for a topology in \( E \).

Note that \( a \) belongs to \( U_a \), and also that if \( a' \) belongs to \( U_a \) then \( U_a = U_{a'} \).

The collection of sets \( U_a \) is indeed a basis for a topology, because if an element \( a \in E \) belongs to two such sets \( (U_1)_{a_1} = (U_1)_a \) and \( (U_2)_{a_2} = (U_2)_a \) then it also belongs to another set \( U_a \) contained in their intersection (just choose a path-connected \( U \) contained in \( U_1 \cap U_2 \) and containing \( p(a) \)), and because the union of them all is \( E \).

The main thing in showing that the projection \( p \) makes \( E \) a covering space of \( X \) is to note that for each good \( U \) and each \( a \) the map \( U_a \to U \) is a homeomorphism. This is straightforward.

Finally, to see that \( E \) is simply connected, just verify that the right action of \( \pi_1(X, x_0) \) on the fiber \( p^{-1}(x_0) \) by path lifting (transport) is free and transitive. This fiber is, by definition of \( E \), the same set as \( p^{-1}(x_0) \). Making sure that the action is the expected one means identifying the (continuous) liftings of loops. If \( \alpha \) is a loop in \( X \) based at \( x_0 \) then one has to see that the following map \( I \to E \) is continuous and therefore is the lifting \( \bar{\alpha} \) guaranteed by Lemma 1:

\[
\begin{align*}
\bar{\alpha} : s &\mapsto [(u \mapsto \alpha(su))].
\end{align*}
\]

To verify continuity at \( s_0 \), just choose a good set \( U \) containing \( p([(u \mapsto \alpha(s_0u))]) = \alpha(s_0) \), note that a neighborhood of \( s_0 \) is mapped into a sheet \( U_a \), and observe that the continuity of the map to \( U_a \) is equivalent to the continuity of the corresponding map to \( U \) (composition with \( p \)), which in this case is simply \( \alpha \) (restricted to that neighborhood of \( s_0 \)).

THE GROUPOID POINT OF VIEW

There is a slightly different way of doing all of this, in which no basepoint \( x_0 \) is singled out and the assumption of path-connectedness is dropped.

The \textit{fundamental groupoid} \( \pi(X) \) is the category whose objects are the points of \( X \), and in which a morphism from \( x \) to \( x' \) is a path class. Transport makes a contravariant functor from \( \pi(X) \) to the category of sets, for every covering space \( (E, p : E \to X) \). It takes the object \( x \) to the fiber \( p^{-1}(x) \), and the morphism \( a \) to the transport map \( (e \mapsto ea) \).

A map of covering spaces gives a (natural) map of functors. We get, in fact, a functor from the category \textit{Cov}_X to the category \textit{Fun}(\pi(X)^\text{op}, \text{Sets}) of functors \( \pi(X)^{\text{op}} \to \text{Sets} \). This is always faithful, full if \( X \) is locally path-connected, and surjective on isomorphism classes if \( X \) is semi-locally simply-connected.

In the case when \( X \) is path-connected and a point \( x_0 \) is chosen, this boils down to the earlier result,
because the groupoid is one in which all objects are isomorphic. In a little more detail: Suppose that $\Gamma$ is a groupoid in which every object is isomorphic to a particular object $x_0$. Let $G$ be the group of morphisms $x_0 \to x_0$ in $\Gamma$. The inclusion of the full subcategory $\{x_0\} \to \Gamma$ is an equivalence of categories, and it follows that the restriction map $Fun(\Gamma^{op}, Sets) \to Fun(\{x_0\}^{op}, Sets)$ is also an equivalence of categories. But $Fun(\{x_0\}, Sets)$ is the same thing as the category of (right) $G$-sets.