EXCISION ESTIMATES FOR SPACES OF DIFFEOMORPHISMS

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Abstract.

[This is not finished. The surgery section needs to be replaced by a reference to known results. The obvious choice is to refer to results, perhaps from Wall's book, about Poincare embeddings, but that would mean introducing Poincare embeddings into this paper, whereas the plan was to defer that to the related paper with Klein ...]

1. Introduction

This paper builds on the results of [4]. We obtain multirelative connectivity statements concerning spaces of diffeomorphisms, deducing these from analogous results in [4] about spaces of homotopy equivalences. The results about diffeomorphisms have consequences for smooth embeddings. The main results are Theorem E.1 and H.1 below. In the course of this introduction we will explain how it implies a number of other results. At the same time we will also discuss a family of slightly stronger statements. These are given here as conjectures, but John Klein and I hope to write down proofs of them soon.

A. Elementary results.

We begin by discussing some elementary facts of the same general type as the statements we are after.

If $p + q < n$ then a smooth embedding of a p-manifold P in an n-manifold N can always be made disjoint from any q-submanifold Q of N by an isotopy. In other words, the pair $(E(P, N), E(P, N - Q))$ of spaces of embeddings is 0-connected. In fact, the pair is $(n - p - q - 1)$-connected. This statement is well known and can be proved by easy general-position arguments ("dimension counting").

In studying the homotopy types of spaces of embeddings it turns out to be very useful to have multirelative statements generalizing the relative statement above. For an elementary example of such a statement, suppose that Q_1 and Q_2 are disjoint submanifolds of N. In the space $E = E(P, N)$, let E_i be the subspace $E(P, N - Q_i)$. The connectivities of the pairs $(E_1, E_1 \cap E_2)$ and $(E_2, E_1 \cap E_2)$ can be found by dimension-counting, and the Blakers-Massey theorem then gives that the homotopy groups of the triad $(E_1 \cup E_2; E_1, E_2)$ must vanish through dimension $2n - 2p - q_1 - q_2 - 2$. Combining this with the fact that (again by dimension-counting) the pair $(E, E_1 \cup E_2)$ is $(2n - 2p - q_1 - q_2 - 1)$-connected, we find that the homotopy groups of the triad $(E; E_1, E_2)$ vanish through dimension $2n - 2p - q_1 - q_2 - 2$. We prefer the slightly stronger statement that the square diagram

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$$\begin{array}{ccc} E(P, N - Q_1 - Q_2) & \longrightarrow & E(P, N - Q_1) \\ \downarrow & & \downarrow \\ E(, N - Q_2) & \longrightarrow & E(P, N) \end{array}$$

is $(2n-2p-q_1-q_2-3)$-cartesian. (It is stronger in that it "contains more π_0 information"; for definitions and basic facts about multirelative connectivity see sections 2 and 3 of [4].)

An easy combination of dimension counting and higher order analogues of the Blakers-Massey theorem yields the generalization below. Suppose that $(Q_1, \dots, Q_r)$ are disjoint submanifolds of N, with dimensions $(q_1, \dots, q_r)$, and for each subset S of $[r] = \{1, \dots, r\}$ let Q_S be the union of Q_i for i in S. The spaces $E(P, N - Q_S)$ form an r-dimensional cubical diagram $E(P, N - Q_\bullet)$.

Proposition A.1. *Let N be an n-dimensional manifold with $r \geq 1$ disjoint submanifolds Q_i of dimension q_i. For any p-dimensional manifold P the cubical diagram $E(P, N - Q_\bullet)$ is $(1 + \sum_i (n - p - q_i - 2))$-cartesian.*

Proof. The statement is true when $r = 1$. Assume by induction that it is true for $r - 1$. We may take $n - p - q_i$ to be at least two for all i. In fact, suppose that (say) $n - p - q_r$ is less than two. Then the quantity $1 + \sum_i (n - p - q_i - 2)$ then becomes bigger when q_r is omitted from $(q_1, \dots, q_r)$. View the r-cubical diagram as a map $E(P, (N - Q_r) - Q_\bullet) \to E(P, N - Q_\bullet)$ between $(r-1)$-cubical diagrams, where the symbol $\bullet$ now represents a subset of $\{1, \dots, r-1\}$. The r-cube will be k-cartesian if the $(r-1)$-cubes are $(k+1)$-cartesian.

Abbreviate $E = E(P, N)$ and $E_S = E(P, N - Q_S)$. By an easy dimension-counting argument, the pair $(E, E_1 \cup \dots \cup E_r)$ is $k_{[r]}$-connected where $k_{[r]} = -1 + \sum_i (n - q_i - p)$. By exactly the same argument, for every nonempty S in $[r]$ the pair

$$(E_{[r]-S}, \cup_{i \in S} E_{([r]-S) \cup i})$$

is k_S-connected where $k_S = -1 + \sum_{i \in S} (n - q_i - p)$. According to 3.3 of [4] (= 2.5 of [3]), the cubical diagram is then k-cartesian where $k + r - 1$ is the minimum of $\sum_\alpha k_{S\alpha}$ over all partitions $\{S_\alpha\}$ of $[r]$. (The assumption $n - p - q_i \geq 2$ more than takes care of the hypothesis of 3.3 that requires k_S to be a monotone function of S.) The minimum is achieved when $[r]$ is partitioned into singletons, so the number k works out to be $1 - 2r + \sum_i (n - q_i - p) = 1 + \sum_i (n - p - q_i - 2)$.

Proposition A.1 is certainly best possible when $r = 1$, and also when $p = 0$. In general it is certainly not. We believe that, as long as $n - p \geq 3$, the cubical diagram $E(P, N - Q_\bullet)$ is $(1 - p + \sum_i (n - q_i - 2))$-cartesian.

B. The main conjecture.

Let us now be a bit more precise about the hypotheses. (We will be even more precise in section 2.) The manifolds P and Q_i are meant to be compact. Each Q_i is transverse to ∂N and may have two boundary parts, the "external" part $\partial_0 Q_i = Q_i \cap \partial N$ and the "internal" part $\partial_1 Q_i$, which meets $\partial_0 Q_i$ at a corner (and is itself transverse to ∂N). Likewise P is allowed to have two boundary parts $\partial_0 P$ and $\partial_1 P$ meeting at a corner. An embedding of $\partial_0 P$ in ∂N, disjoint from $\partial_0 Q_i$ for each i, is given at the outset, and $E(P, N)$ means the space of all embeddings of P in N restricting to that one on $\partial_0 P$ and transverse to ∂N (also transverse to ∂N when restricted to $\partial_1 P$). Some constructions below will introduce additional corners, but ignoring them will cause no trouble.

For a smooth compact manifold triad $(A; \partial_0 A, \partial_1 A)$ the *handle dimension* of A relative to $\partial_0 A$ is the smallest number p such that A can be made from a collar $I \times \partial_0 A$ by attaching handles of index $\leq p$. If A is a submanifold of N then its handle dimension is always meant to be relative to $\partial_0 A = A \cap \partial N$.

In many statements the dimension of P (or of Q_i) can be replaced by the handle dimension. In particular we shall see that the conjecture expressed at the end of section A above is equivalent to:

Main conjecture B.1. *Let N be an n-manifold. Assume that for some $r \geq 1$ there given disjoint compact submanifolds $(B_1, \ldots, B_r)$ such that for each i the handle dimension q_i of B_i satisfies $q_i \leq n-3$. Let $(A; \partial_0 A, \partial_1 A)$ be a smooth compact manifold triad such that A has handle dimension $p \leq n-3$ relative to $\partial_0 A$, and let an embedding of $\partial_0 A$ in ∂N be given, disjoint from every B_i. Then the cubical diagram $E(A, N - B_\bullet)$ is $(1 - p + \sum_i (n - q_i - 2))$-cartesian.*

C. Thickenings.

We look at what happens to a space of embeddings when the embedded manifold is thickened by means of a disk bundle.

The simplest case of this is the statement that, up to homotopy, an embedding of an m-dimensional disk in the interior of a manifold is the same as a tangent m-frame. More precisely, let A be a closed m-dimensional disk and let $P \in A$ be its center point. Let $GE(A, N)$ be the space of germs of embeddings of A in N at P. The canonical map $E(A, N) \to GE(A, N)$ is a fibration. (To make careful sense of this, one can define $E(A, N)$ as a simplicial set whose k-simplices are fiber-preserving embeddings $A \times \Delta^k \to N \times \Delta^k$ and define $GE(A, N)$ analogously using germs along $P \times \Delta^k$.) The fibers are weakly contractible, in other words ∞-connected, essentially because a compactly parametrized family of embeddings of A in N all having the same germ at P can be deformed into a constant family by composing with a suitable isotopy of A in itself that ends in a small neighborhood of P.

The space $GE(A, N)$ is in turn weakly equivalent to the space of tangent m-frames. In fact, the forgetful map from germs to 1-jets is a fibration, and its fibers are affine spaces, contractible by straight-line homotopies.

The same line of argument yields more generally:

Remark C.1. *Let $(P, \partial_0 P, \partial_1 P)$ be a smooth compact manifold triad. Let ξ be a vector bundle over P, let A be its closed disk bundle and $\partial_0 A$ the closed disk bundle of the restriction of ξ to $\partial_0 P$. View $(P, \partial_0 P)$ as embedded in $(A, \partial_0 A)$ by the zero section. Choose an embedding of $\partial_0 A$ in ∂N. Then the restriction map $E(A, N) \to E(P, N)$ is a fibration, and its fiber over any point $f \in E(P, N)$ has the homotopy type of the space of all monomorphisms of vector bundles, fixed on $\partial_0 P$, from ξ to the normal bundle of the embedding f. In particular, if M is a neighborhood of $\partial_0 A$ in N then the diagram*

$$\begin{array}{ccc} E(A, M) & \longrightarrow & E(A, N) \\ \downarrow & & \downarrow \\ E(P, M) & \longrightarrow & E(P, N) \end{array}$$

is ∞-cartesian.

In a similar fashion one can compare $E(P, N)$ (the space of embeddings of P in N with fixed behavior at $\partial_0 P$) with the space $GE(P, N)$ of germs along $\partial_0 P$ of such embeddings. The latter space is weakly contractible, being equivalent to a convex and therefore contractible space of 1-jets, so $E(P, N)$ is weakly equivalent to the fiber of $E(P, N) \to GE(P, N)$. In fact, we are going to be using that fiber as the preferred model for $E(P, N)$.

In the special case where P is a collar on $\partial_0 P$ the fiber is contractible, by composing with an isotopy of P in itself that finishes up in a small neighborhood of $\partial_0 P$. Thus we have:

Remark C.2. *Suppose that the pair $(P, \partial_0 P)$ is diffeomorphic to $(I \times \partial_0 P, 0 \times \partial_0 P)$. Then (for any embedding of $\partial_0 P$ in ∂N) $E(P, N)$ is weakly contractible.*

As essentially a special case of C.2 we have:

Remark C.3. *Suppose that the pair $(P, \partial_0 P)$ is diffeomorphic to $(D^p_+, D^p_+ \cap S^{p-1})$ where D^p_+ is the closed half-disk $\{(x_1, \ldots, x_p) | x_p \geq 0, |x| \leq 1\}$. Then (for any embedding of $\partial_0 P$ in ∂N) $E(P, N)$ is contractible.*

These remarks are useful in manipulating connectivity statements of the kind being discussed here. For example, using C.1 to replace the manifold A by a tubular neighborhood, Conjecture B.1 is equivalent to the special case in which A is n-dimensional. Using C.2 and an easy induction over the number of handles in A ("handle induction" as in section 4 below) this special case is equivalent to the even more special case in which A is a handle:

$$(A, \partial_0 A) = (D^p, \partial D^p) \times D^{n-p}.$$

At the same time, by an even easier argument, it can be assumed that each B_i is also an n-dimensional handle:

$$(B_i, \partial_0 B_i) = (D^{q_i}, \partial D^{q_i}) \times D^{n-q_i}.$$

On the other hand, using C.1 again, this special case of B.1 is equivalent to a different special case, that in which $(A, \partial_0 A) = (D^p, \partial D^p)$ and $(B_i, \partial_0 B_i) = (D^{q_i}, \partial D^{q_i})$. It follows that Conjecture B.1 is equivalent not only to the special case in which the submanifolds all have codimension zero, but also to the variant in which handle dimension is replaced by dimension.

D. Other forms of the conjecture.

There is another way of looking at these conjectures. Consider, for each S, the map from $E(A \cup B_S, N)$ to $E(B_S, N)$ given by restriction; it is a fibration with fiber $E(A, N - B_S)$. Conjecture B.1 is thus equivalent (see 2.4 of [4]) to the statement that the $(r+1)$-cube

$$E(A \cup B_\bullet, N) \longrightarrow E(B_\bullet, N)$$

is $(1 - p + \sum_i (n - q_i - 2))$-cartesian. Cleaning this up by renaming A as one of the B's, we obtain:

Conjecture D.1. *Let N be an n-manifold and let $r \geq 2$. For $1 \leq i \leq r$ let $(B_i; \partial_0 B_i, \partial_1 B_i)$ be a compact manifold triad with handle dimension $q_i \leq n-3$ relative to $\partial_0 B_i$. Assume that the $\partial_0 B_i$ are disjointly embedded in ∂N. Then the cubical diagram $E(B_\bullet, N)$ is $(3 - n + \sum_i (n - q_i - 2))$-cartesian.*

Here is yet another version. Let $N \backslash B$ denote the closure of the complement of the compact codimension zero submanifold B in N.

Conjecture D.2. *Let N be an n-manifold. Let $(M; \partial_0 M, \partial_1 M)$ be a compact manifold triad with $\partial_0 M$ embedded in ∂N. Let $r \geq 2$ and let $B_1, \ldots, B_r$ be codimension-zero submanifolds of M, disjoint from each other and from $\partial_0 M$. Let $q_i \leq n-3$ be the handle dimension of B_i relative to $B_i \cap \partial_1 M$. Then the cubical diagram $E(M \backslash B_\bullet, N)$ is $(3 - n + \sum_i (n - q_i - 2))$-cartesian.*

To deduce D.2 from D.1, fix any point e of the space $E(M_{[r]}, N)$. It suffices to show that the cube formed by the fibers over e of the maps $E(M_S, N) \to E(M_{[r]}, N)$ is $(3 - n + \sum_i (n - q_i - 2))$-cartesian. But this cube is (essentially) $E(B_\bullet, N \backslash e(M_{[r]}))$.

In particular if $m = dim M \leq n-3$ then D.2 says that $E(M \backslash B_\bullet, N)$ is (at least) $(3 - n + r(n - m - 2))$-cartesian. As a sample of the uses of such statements, take $(M; \partial_0 M, \partial_1 M)$ to be the interval $(I, \{0, 1\}, \emptyset)$. (We will actually prove the conjecture in this case.) Thus $E(I, N)$ is the space of all 1-dimensional knots in N with endpoints at two given points in ∂N. Let $B_1, \ldots, B_r$ be disjoint closed interior subintervals of I. The conclusion is that (if $n \geq 4$) this knot space admits an $((r-1)(n-3))$-connected map to a homotopy limit of spaces $E(I \backslash B_S), S \neq \emptyset$. These spaces are individually rather easy to understand; by C.1 and C.2, if $k = card(S) - 1$ then $E(I \backslash B_S)$ is weakly equivalent to the space of all k-tuples of nonzero tangent vectors of N located at k distinct points in N. Since $(r-1)(n-3)$ tends to infinity with r, this has the potential to give a great deal of information about $E(I, N)$.

Of course, to realize this potential one needs to build a suitable apparatus to manipulate those homotopy limits. For one rather general construction of that kind, see [W] and [GW].

E. The main result.

We will not prove the conjecture here, but we will come fairly close in Theorem E.1 below. Note that if $(A; \partial_0 A, \partial_1 A)$ is a compact n-manifold triad and $\partial_0 A$ is embedded in ∂N then the group $Diff(N)$ of diffeomorphisms of N fixed on ∂N acts on $E(A, N)$. If A is a submanifold of N, then there is a preferred point in $E(A, N)$ and its isotropy group is (up to homotopy equivalence) $Diff(N \backslash A)$, where $N \backslash A$ is the closure of the complement of A in N. We have the fibration sequence

$$Diff(N \backslash A) \to Diff(N) \to E(A, N).$$

It must be emphasized that this does not mean that every fiber is the same; some fibers might even be empty.

Theorem E.1. *Let N be a compact n-manifold with disjoint n-submanifolds A, B_1, $\ldots$, $B_r (r \geq 1)$ whose handle dimensions satisfy $p \leq n-3, q_i \leq n-3$. Then the cubical diagram $Diff(N \backslash (B_\bullet \cup A)) \to Diff(N \backslash B_\bullet)$ is $(-p + \sum_i (n - q_i - 2))$-cartesian.*

This would follow easily from B.1. It does not imply B.1 (see the Warning after 2.4 in [4]), but it does imply a "looped version"of B.1:

Corollary E.2. *Let N be an n-manifold with disjoint compact submanifolds $A, B_1, \ldots, B_r (r \geq 1)$ whose handle dimensions satisfy $p \leq n-3, q_i \leq n-3$. Then the cubical diagram $\Omega E(A, N - B_\bullet)$ is $(-p + \sum_i (n - q_i - 2))$-cartesian.*

In fact, E.1 directly implies the case of E.2 in which the submanifolds all have codimension zero; and this implies the general case by C.1. There are also symmetrical versions:

Theorem E.3. *Suppose that N has disjoint submanifolds $B_1, \ldots, B_r (r \geq 2)$ whose handle dimensions satisfy $q_i \leq n-3$. Then the cubical diagram $Diff(N \backslash B_\bullet)$ is $(2 - n + \sum_i (n - q_i - 2))$-cartesian.*

Corollary E.4. *Let N be an n-manifold with disjoint compact submanifolds $B_1, \ldots, B_r (r \geq 2)$ having handle dimensions $q_i \leq n-3$. Then the cubical diagram $\Omega E(B_\bullet, N)$ is $(2 - n + \sum_i (n - q_i - 2))$-cartesian.*

E.3 is just E.1 with the names changed. E.4 follows from E.2 just as D.1 follows from B.1.

F. The proof.

Here is an outline of the proof of E.3 (= E.1). Let $H(N)$ be the space of homotopy equivalences $N \to N$ fixed on the boundary.

The analogue of E.3 for homotopy equivalences instead of diffeomorphisms was proved in [4]. (The condition "handle dimension $\leq p$" clearly implies the condition "homotopy codimension $\geq n - p$" in the sense of [4], because it implies that the pair $(A, \partial_0 A)$ is homotopy equivalent relative to $\partial_0 A$ to a relative CW complex of dimension $\leq p$ and that the pair $(A, \partial_1 A)$ is $(n-p-1)$-connected.) Between the diffeomorphisms and the homotopy equivalences are the so-called block diffeomorphisms, so the problem can be broken up into two parts using the diagram:

$$Diff(N \backslash B_\bullet) \to \widetilde{Diff}(N \backslash B_\bullet) \to H(N \backslash B_\bullet).$$

The discrepancy between $\widetilde{Diff}$ and H is accounted for by surgery theory, and in the multirelative setting there turns out to be no discrepancy at all:

Lemma F.1. *With the hypotheses of Theorem E.3, the $(r+1)$-cubical diagram $\widetilde{Diff}(N \backslash B_\bullet) \to H(N \backslash B_\bullet)$ is ∞-cartesian, provided $n \geq 5$.*

The reason is, briefly, this: Surgery theory says that the difference between homotopy equivalences and block diffeomorphisms is a combination of surgery obstructions and normal invariants. Relative surgery

obstructions for $(N, N\backslash B_1)$ vanish by the "(π,π) theorem", because of the hypothesis "handle dimension $\leq n-3$". Relative normal invariants are the same for $(N, N\backslash B_1)$ as for $(N\backslash B_2, N\backslash(B_1 \cup B_2)$ because normal bordism satisfies excision. This is explained in detail in section 3. The account given there is as direct and geometric as possible, with no exact sequences or L-groups appearing. This is partly in order to make it as accessible as possible to non-experts (such as the author), and partly in order to catch plenty of low-dimensional information. The main ideas here are those that go into the Browder-Casson-Sullivan-Wall applications of surgery to embeddings of codimension ≥ 3 ([W], Chapter 11). Note the restriction $n \geq 5$ in the lemma. Something a little weaker is true for smaller values of n, and that will be good enough.

The discrepancy between $Diff$ and $\widetilde{Diff}$ is accounted for by pseudoisotopy theory. The fact about pseudoisotopy theory that is needed here is the main result of [1], a multirelative generalization of Morlet's disjunction lemma. That result is adapted to the present purpose in sections 4 and 5, yielding:

Lemma F.2. *With the hypotheses of Theorem C.3, the* $(r+1)$*-cubical diagram* $Diff(N\backslash B_\bullet) \to \widetilde{Diff}(N\backslash B_\bullet)$ *is* $(-1+\sum_i(n-q_i-2))$*-cartesian.*

G. A related conjecture.

Given A and N as above, so that $\partial_0 A$ is embedded in ∂N, let $F(A,N)$ be the space of all continuous functions from A to N fixed on $\partial_0 A$. This space contains $E(A,N)$.

Conjecture G.1. *With the same hypothesis as in B.1, the* $(r+1)$*-cube* $E(A, N-B_\bullet) \to F(A, N-B_\bullet)$ *is* $(n-2p-1+\sum_i(n-q_i-2))$*-cartesian.*

This would certainly imply B.1, since the r-cube $F(A,\ N-B_\bullet)$ is $(1-p+\sum_i(n-q_i-2))$-cartesian (see 2.2 of [4]) and $n-2p-1 \geq 1-p$. In fact, it is also possible to deduce G.1 from B.1; see section 6 below.

A weak form of G.1 analogous to A.1 can be proved by elementary means.

By C.1 and C.2, Conjecture G.1 is equivalent to the variant that has dimension instead of handle dimension, and also to the special case in which A and B_i have codimension zero.

H. A related result.

We will prove a weak ("looped") version of G.1. In fact, we will prove something a little better, which is related to E.1 as G.1 is to B.1:

Theorem H.1. *With the same hypothesis as in Theorem E.1, the* $(r+2)$*-cubical diagram*

$$\begin{array}{ccc} Diff(N\backslash(B_\bullet \cup A)) & \longrightarrow & Diff(N\backslash B_\bullet) \\ \downarrow & & \downarrow \\ * & \longrightarrow & F(A, N\backslash B_\bullet) \end{array}$$

is $(n-2p-2+\sum_i(n-q_i-2)$*-cartesian.*

Corollary H.2. *With the same hypothesis, the* $(r+1)$*-cubical diagram* $\Omega E(A, N\backslash B_\bullet) \to \Omega F(A, N\backslash B_\bullet)$ *is* $(n-2p-2+\sum_i)n-q_i-2)$*-cartesian.*

H.1 readily implies E.1. The converse is less obvious, but true (see section 6).

2. Definitions of various function space.

If N is a smooth compact manifold, then $Diff(N)$ will mean the simplicial group of C^∞ self-diffeomorphisms of N relative to the boundary, where "relative to" a given set means agreeing with the identity in a neighborhood of the set Thus a k-simplex of $Diff(N)$ is any diffeomorphism f from $N \times \Delta^k$ to itself relative to $(\partial N) \times \Delta^k$ which is fiber-preserving, in the sense that $p \circ f = p$ where p

is the projection $N \times \Delta^k \to \Delta^k$. Face and degeneracy homomorphisms are defined in the obvious way. One could alsoconsider the topological group $Diff(N)$, putting the usual Whitney topology on the set of diffeomorphisms. The simplicial group $Diff(N)$ is known to be homotopy-equivalent to the singular complex of the topological group. (One can think of it as a kind of "smooth singular complex".)

It is also known that certain other variations in the definition of $Diff(N)$ do not affect the homotopy type. For example, the meaning of "relative to" can be weakened so that f merely fixes the boundary pointwise. (Compare section C above.) One advantage of the chosen model for $Diff(N)$ is that it allows $Diff(M)$ to be identified with a (simplicial) subgroup of $Diff(N)$ whenever M is a compact codimension-zero submanifold of N.

When we say that a manifold A (of any dimension) is a submanifold of N, we will generally intend that the boundary of A is the union of two manifolds $\partial_0 A = A \cap \partial N$ and $\partial_1 A$ (either or both of which may be empty) whose intersection is a corner $\partial\partial_0 A = \partial\partial_1 A$, both A and $\partial_1 A$ being transverse to ∂N. Likewise, when we consider smooth embeddings f of A in N, we will assume that the boundary of A is the union of two manifolds $\partial_0 A$ and $\partial_1 A$ meeting at a corner $\partial\partial_0 A = \partial\partial_1 A$, and we will consider only embeddings such that $f(\partial_0 A) = f(A) \cap \partial N$, and such that both $f(A)$ and $f(\partial_1 A)$ are transverse to ∂N. The symbol $E(A, N)$ will mean the space of embeddings of A in N "relative to $\partial_0 A$". (This is defined as soon as we specify an embedding of A in ∂N, or even the germ of such an embedding along $\partial_0 A$. Up to homotopy, that is the same as specifying an embedding of $\partial_0 A$ in ∂N.) More precisely, $E(A, N)$ is a simplicial set whose k-simplices are the fiber-preserving C^∞ embeddings of $A \times \Delta^k$ in $N \times \Delta^k$ relative to $(\partial_0 A) \times \Delta^k$.

The group $Diff(N)$ acts on $E(A, N)$ by composition. By isotopy extension, the map from $Diff(N)$ to $E(A, N)$ given by acting on any chosen 0-simplex is a fibration. In particular, each orbit of the action is the union of some set of components of $E(A, N)$. In case A has codimension zero, this means that a union of components of $E(A, N)$ is isomorphic to the coset space $Diff(N)/Diff(N\backslash f(A))$ where $f(A)$ is an embedded copy of A in N and $N\backslash f(A)$ is the closure in N of the complement of $f(A)$. Here $Diff(N\backslash A)$ is viewed as a simplicial subgroup of $Diff(N)$. We may say that there is a fibration sequence

$$Diff(N\backslash f(A)) \to Diff(N) \to E(A, N) \tag{2.1}$$

as long as it is understood that this is only a statement about the fiber over $f \in E(A, N)$. (Fibers over other components of $E(A, N)$ might even be empty.)

A concordance of N (also called a pseudoisotopy) is a self-diffeomorphism f of $I \times N$ relative to $0 \times N$ and $I \times \partial N$; thus on $1 \times N$ it induces a self-diffeomorphism f_1 of N relative to ∂N. We define a simplicial group $C(N)$ of concordances of N like our model for $Diff(N)$. A k-simplex of $C(N)$ is a self-diffeomorphism f of $I \times N \times \Delta^k$, fiber-preserving over Δ^k, agreeing with the identity near $0 \times N \times \Delta^k$ and $I \times \partial N \times \Delta^k$, and also agreeing with $id \times f_1$ near $1 \times N \times \Delta^k$ for some self-diffeomorphism f_1 of $B \times \Delta^k$. The simplicial homomorphism $f \mapsto f_1$ from $C(N)$ to $Diff(N)$ is a fibration. There is a fibration sequence

$$Diff(I \times N) \to C(N) \to Diff(N) \tag{2.2}$$

If A is a compact submanifold of N then a concordance embedding of A in N is an embedding of $I \times A$ in $I \times N$ fixing $0 \times A$ and $I \times \partial_0 A$ pointwise and taking $1 \times A$ into $1 \times N$. The simplicial set $CE(A, N)$ of concordance embeddings has as k-simplices those embeddings of $I \times A \times \Delta^k$ in $I \times N \times \Delta^k$ which are fiber-preserving over Δ^k, agree with the inclusion map near $0 \times A \times \Delta^k$ and $I \times \partial A \times \Delta^k$, and agree with $id \times f_1$ near $1 \times A \times \Delta^k$ for some embedding f_1 of $A \times \Delta^k$ in $N \times \Delta^k$. In the codimension-zero case there is a fibration sequence

$$C(N\backslash A) \to C(N) \to CE(A, N) \tag{2.3}$$

analogous to 2.1. This sequence is in fact a little more informative than 2.1 because it happens (see 4.1a below) that in the cases studied here (handle dimension $\leq n - 3$) the space $CE(A, N)$ is connected.

Remark 2.4. If A is a compact tubular neighborhood of P in N, then the restriction map $CE(A,N) \to CE(P,N)$ is not only a fibration (as in Remark C.1) but also a homotopy equivalence. The reason is that each of its fibers is homotopy equivalent to the space of isomorphisms between two vector bundles over $I \times P$ restricting to a given isomorphism on the subspace $(0 \times P) \cup (I \times \partial P)$, which is a deformation retract of $I \times P$.

Remark 2.5. If P is a half-disk, as in Remark C.3, then $CE(P,N)$ is contractible.

We also need the group $\widetilde{Diff}(N)$ of "block diffeomorphisms" of the compact manifold N. A k-simplex is a self-diffeomorphism of $N \times \Delta^k$ relative to $\partial N \times \Delta^k$ that preserves faces, in the sense that for each face σ of Δ^k it restricts to give a diffeomorphism from $N \times \sigma$ to itself. $\widetilde{Diff}(N)$ is not a simplicial group, but only a Δ-group in the sense of [RS]; degeneracy maps are lacking. Of course, $\widetilde{Diff}(N)$ has as a Δ-subgroup the underlying Δ-group of the simplicial group $Diff(N)$.

Finally there is $H(N)$, the space of homotopy equivalences from the compact manifold N to itself fixed on the boundary.

At the core of the proof are maps

$$Diff(N) \to \widetilde{Diff}(N) \to H(N).$$

Most of the time we think of $Diff(N)$ as a simplicial group, $\widetilde{Diff}(N)$ as a Δ-group, and $H(N)$ as a topological monoid. To make sense of the maps above, we consider $Diff(N)$ as a Δ-group by forgetting the degeneracy maps, and also we use a "block" model for $H(N)$ which makes it a Δ-monoid: a k-simplex in this model of $H(N)$ is a self-map of $N \times \Delta^k$, fixed on $\partial N \times \Delta^k$, that preserves faces and that is a homotopy equivalence. This is homotopy equivalent to the underlying Δ-monoid of the simplicial monoid which is the total singular complex of the topological monoid $H(N)$.

3. The surgery step.

We prove Lemma F.1. It is sufficient to consider the case $r = 2$, because (2.2 of [4]) a map between two ∞-cartesian r-cubes is always ∞-cartesian when considered as an $(r+1)$-cube. So let B_1 and B_2 be disjoint codimension-zero submanifolds of N, and assume that the handle dimension of B_1 is at most $n-3$. (The handle dimension of B_2 does not matter here.)

It is also legitimate to replace "homotopy equivalences" by "simple homotopy equivalences" in the statement of the lemma. In fact let $H^s(N)$, a union of components of $H(N)$, be the subspace consisting of simple homotopy equivalences. We have maps of 2-cubes

$$\widetilde{Diff}(N\backslash B_\bullet) \quad \to \quad H^s(N\backslash B_\bullet) \quad \to \quad H(N\backslash B_\bullet).$$

The second map is ∞-cartesian; in fact, both of the squares

$$\begin{array}{ccc} H^s(N\backslash B_{12}) & \to & H(N\backslash B_{12}) \\ \downarrow & & \downarrow \\ H^s(N\backslash B_2) & \to & H(N\backslash B_2) \end{array} \qquad \begin{array}{ccc} H^s(N\backslash B_1) & \to & H(N\backslash B_1) \\ \downarrow & & \downarrow \\ H^s(N) & \to & H(N) \end{array}$$

are pullback squares and therefore ∞-cartesian. The point is that, because the pair $(B_1, \partial_1 B_1)$ is 2-connected, the Whitehead torsion of $f : N\backslash B_1 \to N\backslash B_1$ can be identified with that of $f \cup 1 : N \to N$.

We have to consider the Δ-set $H^s(N)$, together with its subspaces $\widetilde{Diff}(N), H^s(N\backslash B_1)$, and $H^s(N\backslash B_2)$ and their double and triple intersections:

$$\begin{array}{ccc} \widetilde{Diff}(N\backslash B_{12}) & \longrightarrow & H^s(N\backslash B_{12}) \\ \downarrow & & \downarrow \\ \widetilde{Diff}(N\backslash B_1) & \longrightarrow & H^s(N\backslash B_1) \end{array}$$

The task is to show that this diagram is ∞-cartesian.

The proof depends on a lemma from surgery theory, 3.5 below, which in turn depends on the next two standard results, the s-cobordism theorem and the (π,π) theorem:

Theorem 3.2: Let (W,V) be a smooth compact s-cobordism on the pair $(M,\partial M)$, in other words an h-cobordism such that the homotopy equivalence $M \to W$ is simple. Then any product structure $V \cong \partial M \times I$ extends to a product structure $W \cong M \times I$, provided $dim(W) \geq 6$.

Theorem 3.3: Let $(X;Y,Z)$ be a smooth compact manifold triad of dimension ≥ 6, and suppose that the pair (X,Y) is 2-connected. Then every degree-one normal map $(A;B,C) \to (X;Y,Z)$ such that $(C,B\cap C) \to (Z,Y\cap Z)$ is a simple homotopy equivalence of pairs is normally cobordant $(rel\, Z)$ to a simple homotopy equivalence of triads.

(This is a version of Theorem 3.3 of [W]. By definition here a simple homotopy equivalence of triads is a map of triads $(A;B,C) \to (X;Y,Z)$ such that the maps $A \to X, B \to Y, C \to Z$ and $B\cap C \to Y \cap Z$ are simple homotopy equivalences. Likewise, a simple homotopy equivalence of pairs is a map of pairs $(A,B) \to (X,Y)$ such that the maps $A \to X$ and $B \to Y$ are simple homotopy equivalences.)

We need the following variant of 3.3:

Corollary 3.4: Let $(X;Y_1,Y_2,Z)$ be a smooth compact manifold 4-ad of dimension ≥ 7. Suppose that the pairs $(X,Y_1\cup Y_2)$ and $(Y_2,Y_1\cap Y_2)$ are 2-connected. Then every degree-one normal map $(A;B_1,B_2,C) \to (X;Y_1,Y_2,Z)$ such that $(C;B_1\cap C,B_2\cap C) \to (Z;Y_1\cap Z,Y_2\cap Z)$ is a simple homotopy equivalence of triads is normally cobordant $(relZ)$ to a simple homotopy equivalence of 4-ads.

Proof: Apply 3.3 twice, first doing surgery on B_2 away from $B_2\cap C$ and then doing surgery on A away from $B_2\cup C$. In more detail:

First apply 3.3 to the given map of triads

$$(B_2;B_1\cap B_2,B_2\cap C) \to (Y_2;Y_1\cap Y_2,Y_2\cap Z).$$

The result is a certain normal cobordism of triads over $(Y_2;Y_1\cap Y_2,Y_2\cap Z)$, say $(V;W,P)$, between $(B_2;B_1\cap B_2,B_2\cap C)$ and, say, $(B_2';W\cap B_2',P\cap B_2')$, where $(P,W\cap P)$ is a trivial cobordism (over $(Y_2\cap Z,Y_1\cap Y_2\cap Z)$) on $(B_2\cap C,B_1\cap B_2\cap C)$ and the map $(B_2',W\cap B_2') \to (Y_2,Y_1\cap Y_2)$ is a simple homotopy equivalence of pairs.

Attach V to A along B_2. There is then a normal cobordism of 4-ads (over $(X;Y_1,Y_2,Z)$) between $(A;B_1,B_2,C)$ and $(A\cup V;B_1\cup W,B_2'\cup P,C)$. In fact, $((A\cup V)\times I;(B_1\cup W)\times I,(V\times 0)\cup(B_2'\times I)\cup(P\times I),C\times I)$ serves as a normal cobordism (of 4-ads, $rel\, Z$) between $A = A\times 0$ and $A\cup V = (A\cup V)\times 1$.

We conclude that without loss of generality the map $(B_2,B_1\cap B_2) \to (Y_2,Y_1\cap Y_2)$ is a simple homotopy equivalence of pairs. To complete the proof apply 3.3 to the map of triads

$$(A;B_1,B_2\cup C) \to (X;Y_1,Y_2\cup Z)$$

Lemma 3.5: Let $(X;Y_1,Y_2,Z)$ be a smooth compact manifold 4-ad of dimension ≥ 7. Suppose that

(i) the pair $(X,Y_1\cup Y_2)$ is ∞-connected, and

(ii) the pair $(Y_2,Y_1\cap Y_2)$ is 2-connected.

If a 4-ad map $h:(X;Y_1,Y_2,Z) \to (X;Y_1,Y_2,Z)$ is a simple homotopy equivalence on each of X,Y_1,Y_2, and $Y_1\cap Y_2$ and a diffeomorphism on each of $Z,Y_1\cap Z,Y_2\cap Z$, and $Y_1\cap Y_2\cap Z$, then h is homotopic, as a map of 4-ads, by a homotopy fixed on Z, to a diffeomorphism. If in addition h maps a neighborhood of Z diffeomorphically to a neighborhood of Z, then the homotopy may be chosen to be fixed near Z.

Proof of 3.5: Let Y be $Y_1\cup Y_2$. We have a map of triads $h:(X;Y,Z) \to (X;Y,Z)$ which is a simple homotopy equivalence on X and on Y and a diffeomorphism on Z and on $Y\cap Z$. Consider it as a degree-one normal map. The first step, which uses hypothesis (i), is to make a normal cobordism of triads

$$(U;V,Z\times I) \to (X;Y,Z)$$

between h and the identity that is in a certain sense trivial over Z. In more detail, write

$$Z' = (X \times 0) \cup (Z \times I) \cup (X \times 1)\, and$$
$$W = (X \times 0) \cup (Z \times I) \cup_h (X \times 1).$$

(In W the point $(z, 1) \in Z \times I$ is identified with $(h(z), 1) \in X \times 1$ for every z in Z.) Let $f : W \to Z'$ be given by

$$f(x, 0) = (h(x), 0)$$
$$f(z, t) = (h(z), t), and$$
$$f(x, 1) = (x, 1).$$

The assertion is that there is a degree-one normal map of triads

$$f : (U; V, W) \to (X \times I; Y \times I, Z'),$$

for some U and V, such that on Wf is given by the formulas above. In fact, the manifold U may be taken to be a trivial cobordism on W. The given map of pairs $f : (W, \partial W = V \cap W) \to (Z', \partial Z')$ can be extended to a map of triads because of (i). The bundle data making f a normal map can be extended from Z' to $X \times I$, again because of (i). "Degree one" is automatic.

The next step is to improve f so as to make it a degree-one normal map of 4-ads

$$f : (U; V_1, V_2, W) \to (X \times 1; Y_1 \times 1, Y_2 \times 1, Z')$$

instead of triads. This is a straightforward transversality argument: The given map $V \to Y \times 1$ (a rest riction of f) can be altered, by a homotopy which is fixed on $V \cap W$, so that $f^{-1}((Y_1 \cap U_2) \times 1)$ is a smooth manifold separating V into manifolds $V_i = f^{-1}(Y_i \times 1), i = 1, 2$, and then this homotopy on V, together with the constant homotopy on W, can be extended to a homotopy of f on all of U.

Now alter f, by a normal cobordism of 4-ads $rel\, Z'$, so as to make it a simple homotopy equivalence. (This uses 3.4, whose hypotheses follow from (i) and (ii).)

At this stage U, V_1, V_2, and $V_1 \cap V_2$ are s-cobordisms on X, Y_1, Y_2, and $Y_1 \cap U_2$. Apply 3.2 to each of them in turn, starting with $V_1 \cap V_2$ and ending with U. The result is a diffeomorphism $\Phi : U \to X \times I$ which is the identity on $X \times 0$ and on $Z \times I$, and which carries V_i to $Y_i \times 1$. The composition of f with Φ^{-1} gives the desired homotopy.

The final statement in the lemma can be proved using a boundary collar.

To show that 3.1 is ∞-cartesian it will suffice to prove the following statement:

Claim: Let $k \geq 2$, and suppose that h is a k-simplex of $H^s(N)$ such that the faces d_0H, d_1H, and d_2h belong respectively to $\widetilde{Diff}(N), H^s(N \backslash B_1)$, and $H^s(N \backslash N_2)$, and such that any remaining faces $d_jh\,(2 < j \leq k)$ are identity maps. Then there is a $(k+1)$-simplex H of $H^s(N)$ such that $d_0H = h$, and such that d_1H, d_2H, and d_3H belong respectively to $\widetilde{Diff}(N), H^s(N \backslash B_1)$, and $H^s(N \backslash B_2)$, and such that any remaining faces $d_jh\,(3 < j \leq k+1)$ are identity maps.

The claim suffices because of the fact that all eight of the Δ-sets in 3.1 satisfy the Kan extension condition: The desired conclusion is that a certain map $(*)$ from $\widetilde{Diff}(N \backslash B_{12})$ to the homotopy limit of the other seven spaces is ∞-connected. This amounts to the statement that for each $k \geq 3$ and for each vertex of $\widetilde{Diff}(N \backslash B_{12})$ the homotopy set $\pi_k(H^s(N); \widetilde{Diff}(N), H^s(N \backslash B_1), H^s(N \backslash B_2)$ of the 4-ad has exactly one element, together with the statement that the map $(*)$ induces a surjection on π_0. For the latter we use the case $k = 2$ of the claim: The Kan condition implies that every component of the homotopy limit can be represented by a 2-simplex h of $H^s(N)$ whose three faces d_0h, d_1h, and d_2h belong respectively to $\widetilde{Diff}(N), H^s(N \backslash B_1)$, and $H^s(N \backslash B_2)$. Then the claim shows that that component is hit

by some component of $\widetilde{Diff}(N\backslash B_{12})$. The case $k \geq 3$ of the claim shows (using the Kan condition again) that π_k of the 4-ad vanishes when the basepoint is taken to be the identity. Since (3.1) is a diagram of homomorphisms, the same is then true with any vertex of $\widetilde{Diff}(N\backslash B_{12})$ as basepoint.

Proof of Claim The given element h is a simple homotopy equivalence from $N \times \Delta^k$ to itself satisfying the following boundary conditions (a), (b), and (c). Let $d^i : \Delta^{k-1} \to \Delta^k$ be the ith face inclusion. Then h

(a) maps $N \times d^0\Delta^{k-1}$ to itself by a diffeomorphism, and
(b) maps $N\backslash B_j \times d^i\Delta^{k-1}$ to itself by a simple homotopy equivalence (for $i = 1, 2$), and
(c) coincides with the identity in the set

$$B = ((\partial N) \times \Delta^k) \cup (B_1 \times d^1\Delta^{k-1}) \cup (B_2 \times d^2\Delta^{k-1}) \cup (N \times \cup_{2<j\leq k)} d^j\Delta^{k-1}).$$

Altering h by a small homotopy, we can assume that h

(c′) coincides with the identity in a neighborhood of B,

and that the behavior of h at $N \times d^0\Delta^{k-1}$ determines its behavior in a neighborhood of $N \times d^0\Delta^{k-1}$ by the rule:

(d) $h(x, u) = (x', u') \to h(x, (1 - s)v + su)) = (x, (1 - s)v + su')$.

Here x and x' are points in N, u and u' are points in $d^0\Delta^{k-1}$, v is the vertex of Δ^k opposite $d^0\Delta^{k-1}$, and s ranges over the interval $(1 - \epsilon, 1)$ for some ϵ independent of x and u.

We now apply 3.5, with $X = N \times \Delta^k$, $Y_i = (N\backslash B_i)\times d^i\Delta^{k-1}$, and $Z = B\cup(N \times d^0\Delta^{k-1})$. This yields a homotopy h_t from $h = h_0$ to a diffeomorphism h_1. The homotopy satisfies certain conditions. In particular each h_t satisfies the boundary conditions $(a), (b), (c')$, and (d). Writing $H^*(x, u, t)$ for $h_t(x, u,), t)$, we have a simple homotopy equivalence H^* from $N \times \Delta^k \times I$ to itself satisfying certain conditions.

What we wanted was a simple homotopy equivalence H from $N \times \Delta^{k+1}$ to itself satisfying certain conditions. To obtain H from H^*, define $p : N \times \Delta^k \times I \to N \times \Delta^{k+1}$ by $p(x, u, t) = (x, (1 - t)d^0(u) + td^1(u))$, and put $H \circ p = p \circ H^*$. The map H is then well-defined (even though p is not injective) because of the fact that h_t is constant on $N \times d^0\Delta^{k-1}$. The desired conditions for H follow. The significance of (d) is that it makes H act like a diffeomorphism on the sets where it is intended to do so.

We note for future reference that the proof of the Claim did not require $n \geq 5$ and $k \geq 2$; it only required $n + k \geq 7$. It follows that even if $n = 4$ the map $(^*)$ still induces an injection on π_0 and isomorphisms on all $\pi_i, i > 0$.

4. The multirelative disjunction lemma.

The next lemma is essentially the main result of [1], and it will play a key role in proving Lemma F.2. The main business of this section is to deduce Lemma 4.1 from what was actually proved in [1].

Lemma 4.1 is an analogue of Conjecture B.1 for concordance embeddings; note, however, that it gives a much higher connectivity. The case $r = 1$ of 4.1 is (essentially) Morlet's Disjunction Lemma ([BLR]).

Lemma 4.1. *If $A, B_1, \ldots, B_r, r \geq 1$, are disjoint compact codimension-zero submanifolds of the manifold N with handle dimensions $p, q_1, \ldots, q_r$ all $\leq n - 3$, then the r-cubical diagram $CE(A, N\backslash B_\bullet)$ is $(n - p - 2 + \sum_i(n - q_i - 2))$-cartesian.*

Lemma 4.1 is true in the case $r = 0$ as well. Since a k-cartesian 0-cubical diagram is the same as a $(k - 1)$-connected space, the statement is:

Lemma 4.1a. *If A is a compact codimension-zero submanifold of N with handle dimension $p \leq n - 3$, then the space $CE(A, N)$ is $(n - p - 3)$-connected.*

There are also the related statements:

Lemma 4.1*. *If $P, Q_1, \dots, Q_r, r \geq 1$, are disjoint compact submanifolds of the manifold N with dimensions $p, q_1, \dots, q_r$ all $\leq n-3$, then the r-cubical diagram $CE(P, N - Q_\bullet)$ is $(n-p-2+\sum_i(n-q_i-2))$-cartesian.*

Lemma 4.1a*. *If P is a compact submanifold of N with dimension $p \leq n-3$, then the space $CE(P, N)$ is $(n-p-3)$-connected.*

We will deduce 4.1*from [1] and then deduce the other three statements above from it (although in fact 4.1a is older than 4.1*[H]).

For 4.1*, use 3.3 of [4] exactly as in the proof of Proposition A.1 above. It is enough if the pair $(CE(P, N), \cup_i CE(P, N - Q_i))$ is $(n-p-2+\sum_i(n-q_i-2))$-connected. This is exactly what is proved in [1], statement (186), page 253. (In [1] a cruder "higher Blakers-Massey" result was used to pass from (186) to the vanishing of $(r+1)$-ad homotopy groups, a statement slightly weaker than the one obtained in 4.1*.)

Remark 4.2. In view of 2.4, 4.1* implies some cases of 4.1, namely those cases in which A and B_i are tubular neighborhoods of manifolds P and Q_i of the kind considered in 4.1*. Also by 2.4 4.1a* follows from 4.1a.

Two induction devices are involved in obtaining 4.1 and 4.1a from 4.1*. The first is quite elementary. I learned the second long ago from [BLR] (pp. 23-24) (It seems that I have been using it or something like it ever since.) Some of the arguments below have appeared in some form in [2].

First device (handle induction).

Suppose that A is a compact codimension-zero submanifold of N and that H is a compact codimension-zero submanifold of $N \backslash A$. For example, H might be a handle attached to A. There is a fibration sequence

$$CE(H, N\backslash A) \to CE(A \cup H, N) \to CE(A, N).$$

One consequence is that $CE(A \cup H, N)$ is k-connected as soon as both $CE(A, N)$ and $CE(H, N\backslash A)$ are k-connected. This reduces the proof of 4.1a to the case when A is a single handle of index $p, (A, \partial_0 A) = (D^p, \partial D^p) \times D^{n-p}$.

Similarly 4.1 can be reduced to the case in which A is a single handle: If the r-cube $CE(A, N\backslash B_\bullet)$ is known to be k-cartesian, then for $CE(A \cup H, N\backslash B_\bullet)$ to be k-cartesian it will be enough if the map $CE(A \cup H, N\backslash B_\bullet) \to CE(A, N\backslash B_\bullet)$ is a k-cartesian $(r+1)$-cube, and for this it will be enough (by 2.4) if the r-cube of fibers $CE(H, N\backslash A\backslash B_\bullet)$ is k-cartesian. Actually, in applying 2.4, we should consider fibers over any possible point in $CE(A, N\backslash(B_1 \cup \dots \cup B_r)$, so this step depends on the fact that the latter space is connected (a consequence of 4.1a, the proof of which will be completed shortly).

We can further reduce 4.1 to the case in which each $B_i, 1 \leq i \leq r$, is also a single handle. For example, in the case $r = 1$, if each of the two inclusion maps

$$CE(A, N\backslash(B \cup H)) \to CE(A, N\backslash B) \to CE(A, N)$$

is k-connected then so is their composition.

Thus, using 4.1a, we have reduced the proof of 4.1 to cases in which it follows (see remark 4.2) from the known result 4.1*.

Second device (handle splitting).

Suppose that $A \cong D^{n-p} \times D^p$ is a handle of index p in N (meeting ∂N transversely in $\partial_0 A \cong D^{n-p} \times S^{p-1}$). If $p > 0$, then the p-disk can be cut into three pieces by two parallel planes. Forming the product of D^{n-p} with each piece, write A as $A^+ \cup A^0 \cup A^-$. The middle piece, A^0, is a handle of index

$p-1$ in N; the other two, A^+ and A^-, are disjoint handles of index p in $N\backslash A^0$. Consider the square diagram of restriction maps

$$\begin{array}{ccc} E(A,N) & \longrightarrow & E(A^+\cup A^0,N) \\ \downarrow & & \downarrow \\ E(A^0\cup A^-,N) & \longrightarrow & E(A^0,N). \end{array}$$

The upper right and lower left spaces are contractible, by C.3. All four arrows in the square are fibrations. Taking fibers of the vertical maps yields the inclusion map

$$E(A^+,N\backslash(A^0\cup A^-))\to E(A^+,N\backslash A^0).$$

This is a map from a space having the homotopy type of $E(A,N)$ to a space having the homotopy type of $\Omega E(A^0,N)$, where A and A^0 are handles in N with respective indices p and $p-1$.

Every statement in the paragraph above remains true if spaces of embeddings are replaced by spaces of concordance embeddings.

We can now complete the proof of 4.1a. We must prove that $CE(A,N)$ is $(n-p-3)$-connected when A is a handle of index $p\le n-3$ in N; we do so by induction on p. Given such a handle with $p>0$, choose A^0, and A^- as above. We have (essentially) a map $CE(A,N)\to\Omega CE(A^0,N)$. By induction on $p, CE(A^0,N)$ is $(n-p-2)$-connected and so $\Omega CE(A^0,N)$ is $(n-p-3)$-connected. The map itself is $(2n-2p-4)$-connected by a proved case of 4.1. Since $2n-2p-4>n-p-3$, the space $CE(A,N)$ is $(n-p-3)$-connected.

To begin the induction let A be a handle of index zero, in other words a closed tubular neighborhood of an interior point P in N. By dimension-counting, $CE(P,N)$ (which by 2.4 is homotopy-equivalent to $CE(A,N)$) is $(n-3)$-connected. In fact, $CE(P,N)$ is contained in a contractible space, call it $CF(P,N)$: the space of all smooth maps from $P\times I$ to $N\times I$ taking $P\times 0$ to $P\times 0$, $P\times I$ into $int\,N\times I$, and $P\times int\,I$ into $N\times int\,I$, and transverse to $P\times\partial I$. Dimension counting shows that the pair $(CF(P,N),CE(P,N))$ is $(n-2)$-connected, so that $CE(P,N)$ is $(n-3)$-connected.

5. The pseudoisotopy step.

We prove Lemma F.1. The proof works even if $r=1$. Our starting point is 4.1. We first convert it into a statement, 5.1, which is about concordances instead of concordance embeddings.

With the hypothesis and notations of 4.1, we have for each S an injection of simplicial groups

$$C(N\backslash(A\cup B_S))\to C(N\backslash B_S).$$

The space $CE(A,N\backslash B_S)$ is isomorphic to the coset space $C(N\backslash B_S)/C(N\backslash(A\cup B_S))$ (remember that it is connected, by 4.1a) and therefore it is homotopy equivalent to the homotopy fiber of the map of classifying spaces

$$BC(N\backslash(A\cup B_S))\to BC(N\backslash B_S).$$

All of this is natural in S. We conclude by 4.1 that for the map of r-cubes $BC(N\backslash(A\cup B_\bullet))\to BC(N\backslash B_\bullet)$ of based spaces the r-cube of homotopy fibers is $(n-p-2+\sum_i(n-q_i-2))$-cartesian. This implies (by 2.4 of [4]) that the map itself, viewed as an $(r+1)$-cube, is $(n-p-2+\sum_i(n-q_i-2))$-cartesian. Placing A on an equal footing with the B_i's, we obtain a more symmetrical statement:

Lemma 5.1. *Let $B_1,\dots,B_r, r\ge 1$, be disjoint codimension-zero submanifolds of N with handle dimensions $q_i\le n-3$. Then the r-cube $BC(N\backslash B_\bullet)$ is $\sum_i(n-q_i-2)$-cartesian.*

The next task is to convert 5.1 into a statement, 5.4, that compares diffeomorphisms with block diffeomorphisms. Therefore for the present discussion $Diff(-)$ and $C(-)$ are to be Δ-groups rather than simplicial groups.

For any N there is a commutative diagram of Δ-groups

$$\begin{array}{ccccc} Diff(I\times N) & \longrightarrow & C(N) & \longrightarrow & Diff(N) \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{Diff}(I\times N) & \longrightarrow & \widetilde{C}(N) & \longrightarrow & \widetilde{Diff}(N) \end{array} \tag{5.2}$$

in which the rows are fibration sequences. The upper row is 2.2. The lower row is defined similarly. The new object $\widetilde{C}(N)$ is defined by changing "fiber-preserving" to "face-preserving"in the definition of $C(N)$. The vertical maps in 5.2 are inclusions.

It is not hard to see that the homotopy groups of $\widetilde{C}(N)$ are trivial (see 2.1 of [BLR]). Thus $\pi_i\,\widetilde{Diff}(N)$ is isomorphic to $\pi_{i-1}\widetilde{Diff}(I\times N)$ and, by induction, to $\pi_0\,\widetilde{Diff}(I^i\times N)$, the group of pseudoisotopy classes of diffeomorphisms of $I^i\times N$ fixed on the boundary.

Let $R(N)$ be the coset space $\widetilde{Diff}(N)/Diff(N)$, a Δ-set. It is connected, since it has only one vertex. We claim that the sequence of coset spaces

$$R(I\times N)\to\widetilde{C}(N)/C(N)\approx BC(N)\to R(N) \tag{5.3}$$

is again a fibration sequence – that is, that the map $R(I\times N)\to fiber(u)$ induces an isomorphism on π_i for all $i\geq 0$. Except for π_0 this is immediate, because the fibers of the vertical maps in 5.2 form a fibration sequence which may be identified with the loop space of 5.3. Therefore the point to check is that $fiber(u)$ is connected or, what is the same, that u maps $\pi_1BC(N)=\pi_0C(N)$ onto $\pi_1R(N)$. This is readily checked by going to the definitions: inspection shows that the map $\pi_0C(N)\to\pi_1R(N)$ in question is given by an obvious rule that sends a 0-simplex of $C(N)$ (a certain kind of diffeomorphism of $I\times N$ to a 1-simplex of $\widetilde{Diff}(N)$ (a certain kind of diffeomorphism of $N\times\Delta^1$).

Notice that R is a functor with respect to codimension-zero embeddings, and that the maps in the diagrams above are natural with respect to these.

Lemma 5.4. *With the hypothesis of 5.1, the r-cube $R(N\backslash B_\bullet)$ is $\sum_i(n-q_i-2)$-cartesian.*

Proof. We use induction on r, and for fixed r we show by induction on k that, for $0\leq k\leq\sum_i(n-q_i-2)$ the cube is k-cartesian.

To begin the induction on k: If $r=1, R(N\backslash B_\bullet)$ is a map of connected spaces and therefore 0-connected. If $r>1, R(N\backslash B_\bullet)$ can be viewed as a map of $(r-1)$-cubes which, by induction on r, are 1-cartesian. This makes it 0-cartesian.

For $k\geq 1$ we use 5.3. Note that $R(N\backslash B_\bullet)$ is a cube of based connected spaces, and is known by induction on k to be 0-cartesian. Therefore in order for it to be k-cartesian it will be enough (by the lemma below) if $\Omega R(N\backslash B_\bullet)$ is $(k-1)$-cartesian. By 5.3 the latter is the fiber of a map of cubes:

$$R(I\times(N\backslash B_\bullet))\to\widetilde{C}(N\backslash B_\bullet)/C(N\backslash B_\bullet).$$

The cube $R(I\times(N\backslash B_\bullet))$ is $(k-1)$-cartesian by induction on k. The cube $\widetilde{C}(N\backslash B_\bullet)/C(N\backslash B_\bullet)$ is k-cartesian by 5.1.

We used the following lemma about cubical diagrams of spaces. The proof is left to the reader. (The main point is that loop space commutes with homotopy inverse limit.)

Lemma 5.5. *Let $X_\bullet$ be an r-cube of spaces, and suppose that for every choice of basepoint the cube $\Omega X_\bullet$ is $(k-1)$-cartesian. Then $X_\bullet$ is k-cartesian if the associated map $X_{[r]}\to\underset{S\neq[r]}{holim}\,X_S$ induces a bijection*

on π_0. Thus in particular, $X_\bullet$ is k-cartesian if $X_\bullet$ is 1-cartesian, or if $X_\bullet$ is 0-cartesian and the space $X_{[r]}$ is 0-connected.

Since $R(N\backslash B_S)$ is naturally homotopy equivalent to the homotopy fiber of

$$BDiff(N\backslash B_S) \longrightarrow B\widetilde{Diff}(N\backslash B_S),$$

5.4 implies that, with the same hypothesis, the map of r-cubes $BDiff(N\backslash B_\bullet) \to B\widetilde{Diff}(N\backslash B_\bullet)$ is a $\sum_i(n-q_i-2)$-cartesian $(r+1)$-cube. The conclusion of Lemma F.2 now follows by looping.

Theorem C.3 has now been proved when $n \geq 5$. In the remaining cases we will in fact do better and prove Conjecture B.1.

If $n \leq 3$ then necessarily $n = 3$ and $p = 0$. Thus B.1 follows from A.1.

In treating the case $n = 4$ it will be good to have a name for a condition slightly weaker than "k-cartesian". Call a map of spaces almost k-connected if each of its homotopy fibers is either empty or $(k-1)$-connected. Call a cubical diagram $X_\bullet$ of spaces almost k-cartesian if the associated map $a(X_\bullet)$ is almost k-connected.

Although we were not able to establish Lemma F.1 in the case $n = 4$, we did find that in that case the cube

$$\widetilde{Diff}(N\backslash A_\bullet) \longrightarrow H(N\backslash A_\bullet)$$

is almost ∞-cartesian. By F.2 it follows that the cube

$$Diff(N\backslash A_\bullet) \longrightarrow H(N\backslash A_\bullet)$$

is almost $\sum_i(n-q_i-2)$-cartesian. Since $H(N\backslash A_\bullet)$ is $(2-n+\sum_i(n-q_i-2))$-cartesian, this makes $Diff(N\backslash A_\bullet)$ almost $(2-n+\sum_i(n-q_i-2))$-cartesian. Therefore, given the hypothesis of B.3, the r-cube $\Omega E(A, N\backslash B_\bullet)$ is almost $(-p+\sum_i(n-q_i-2))$-cartesian.

In fact $E(A, N\backslash B_\bullet)$ is $(1-p+\sum_i(n-q_i-2))$-cartesian. This uses the following variant of Lemma 5.5, which we again leave to the reader. Note that the cube $E(A, N\backslash B_\bullet)$ is 1-cartesian by Proposition A.1, since $n-p-q_i-2 = (n-p)+(n-q_i)-6 \geq 0$ for all i.

Lemma 5.6. *If a cubical diagram is 1-cartesian and if after looping (for any basepoint) it becomes almost $(k-1)$-cartesian, then it is k-cartesian.*

Remark 5.7. In some cases Conjecture B.1 follows from the combination of A.1 and E.1. For example, by 5.5 this is so whenever $\sum_i(n-p-q_i-2) \geq 0$. In particular, Conjecture B.1 is true in all cases where $p = 1$.

Remark 5.8. The methods of this paper actually prove something a bit more general than Theorem E.1. The generalization involves looking at diffeomorphisms between distinct manifolds rather than between a manifold and itself. Let $(N, B_1, \ldots, B_r)$ be as in Theorem E.1 and let $(N', B'_1, \ldots, B'_r)$ be another such collection. Suppose given a diffeomorphism $\partial N \cong \partial N'$ and for each i a diffeomorphism $B_i \cong B'_i$ agreeing with it on $\partial_0 B_i$. Write $Diff(N\backslash B_S, N'\backslash B'_S)$ for the space of diffeomorphisms $N\backslash B_S \to N'\backslash B'_S$ with fixed boundary behavior. In this situation the cubical diagram $Diff(N\backslash B_\bullet, N'\backslash B'_\bullet)$ is $(2-n+\sum_i(n-q_i-2))$-cartesian. We sketch the proof. The analogue for homotopy equivalences is known ([4], Remark 1.9). The analogue for block diffeomorphism holds as soon as a corresponding generalization of Lemma F.1 is known, and the proof of F.1 really does generalize to handle this. To pass from block diffeomorphisms to diffeomorphisms is the easiest step. In fact, if the space $Diff(N\backslash B_{[r]}, N'\backslash B'_{[r]})$ is nonempty then the map

$$Diff(N\backslash B_\bullet, N'\backslash B'_\bullet) \longrightarrow \widetilde{Diff}(N\backslash B_\bullet, N'\backslash B'_\bullet)$$

is isomorphic to the map

$$Diff(N\backslash B_\bullet) \longrightarrow \widetilde{Diff}(N\backslash B_\bullet)$$

and so is $(-1+\sum_i(n-q_i-2))$-cartesian by F.2; and otherwise it is trivially ∞-cartesian.

6. Theorem E.1 implies theorem H.1

This is proved by using the same pair of induction devices that appeared in section 4.

We first use handle induction to reduce to the case in which A is a handle, as follows: To obtain the conclusion for $A \cup H$ from the conclusion for A, consider the diagram of r-cubes:

$$\begin{array}{ccccc} Diff(N\backslash(A\cup H\cup B_\bullet)) & \longrightarrow & Diff(N\backslash(A\cup B_\bullet)) & \longrightarrow & Diff(N\backslash B_\bullet) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & F(,N\backslash(A\cup B_\bullet)) & \longrightarrow & F(A\cup H, N\backslash B_\bullet) \\ & & \downarrow & & \downarrow \\ & & * & \longrightarrow & F(A, N\backslash B_\bullet). \end{array}$$

By inductive hypothesis the large right-hand rectangle is an $(n-2p-2+\sum_i(n-q_i-2))$-cartesian $(r+2)$-cube. The same holds for the lower right square. (In fact that square is $(1-p+\sum_i(n-q_i-2))$-cartesian; this comes down to the fact that the map $F(H, N\backslash(A\cup B_\bullet)) \to F(H, N\backslash B_\bullet)$ is $(1-p+\sum_i(n-q_i-2))$-cartesian, which follows from 3.2 of [4].) Therefore the same holds for the upper right square. By the case in which A is a handle it also holds for the upper left square. Therefore it holds for the large upper rectangle.

In the case when A is a single handle, say of index p, we use handle splitting to reduce to the case $p = 0$. Let $p > 0$, and suppose the result known for $p-1$. Cutting up A as in section 4, we have a square of squares of r-cubes:

$$\begin{array}{ccc} X(A) & \longrightarrow & X(A^+\cup A^0) \\ \downarrow & & \downarrow \\ X(A^0\cup A^-) & \longrightarrow & X(A^0) \end{array}$$

with $X(-)$ being shorthand for

see exhibit B

The squares $X(A^+\cup A^0)$ and $X(A^0\cup A^-)$ are ∞-cartesian by Remark C.3. The square $X(A^0)$ is $[n-2(p-1)-2+\sum_i(n-q_i-2)]$-cartesian, by inductive hypothesis, because A^0 is a $(p-1)$-handle. Therefore, by 2.5 of [4], $X(A)$ will be $[n-2p-2+\sum_i(n-q_i-2)]$-cartesian if the entire square of squares is an $[n-2p-2+\sum_i(n-q_i-2)]$-cartesian $(4+r)$-cube.

To see that it is, we use 2.2 of [4] (three times), observing that the square

$$\begin{array}{ccc} Diff(N\backslash(A\cup B_\bullet)) & \longrightarrow & Diff(N\backslash(A^+\cup A^0\cup B_\bullet)) \\ \downarrow & & \downarrow \\ Diff(N\backslash(A^0\cup A^-\cup B_\bullet)) & \longrightarrow & Diff(N\backslash(A^0\cup B_\bullet)) \end{array}$$

is an $(n-2p-2+\sum_i(n-q_i-2))$-cartesian $(2+r)$-cube by Theorem E.1 while the parallel square

$$\begin{array}{ccc} F(A, N\backslash B_\bullet) & \longrightarrow & F(A^+\cup A^0, N\backslash B_\bullet) \\ \downarrow & & \downarrow \\ F(A^0\cup A^-, N\backslash B_\bullet) & \longrightarrow & F(A^0, N\backslash B_\bullet) \end{array}$$

is more or less trivially ∞-cartesian and the other two squares parallel to these are even more trivially ∞-cartesian.

When $p = 0$ (so that A is a tubular neighborhood of a point), the conclusion is trivially true.

By the same method we now show that Conjecture B.1 implies Conjecture G.1. We may assume that A and B_i have dimension n.

First, by induction on the number of handles, we reduce to the case in which A is a handle. To obtain the conclusion for $A \cup H$ from the conclusion for A, consider the diagram (an $(r+2)$-cube displayed as a square of r-cubes):

$$\begin{array}{ccc} E(A \cup H, N - B_\bullet) & \longrightarrow & F(A \cup H, N - B_\bullet) \\ \downarrow & & \downarrow \\ E(A, N \quad -B_\bullet) & \longrightarrow & F(A, N - B_\bullet). \end{array}$$

By assumption the lower arrow is an $(n - 2p - 1 + \sum_i (n - q_i - 2))$-cartesian $(r+1)$-cube. To prove the same for the upper arrow it will be enough to show that the fibers of the vertical maps form an $(n-2p-1+\sum_i(n-q_i-2))$-cartesian $(r+1)$-cube themselves (for every choice of point in $E(A, N - B_{[r]})$).

This means examining the composed map

$$E(H, (N \backslash A) - B_\bullet)) \longrightarrow F(H, (N \backslash A) - B_\bullet) \longrightarrow F(H, N - B_\bullet)$$

Here the first arrow is $(n-2p-1+\sum_i(n-q_i-2))$-cartesian if the result is true for a single handle. The second is just as good: by 3. of [4] it is $(1 - p + (n - p - 2) + \sum_i (n - q_i - 2))$-cartesian.

In the case when A is a single handle, say of index p, we use handle splitting to reduce to the case $p = 0$. Let $p > 0$, and suppose the result known for $p - 1$. Cutting up A as usual, we have a square diagram

$$\begin{array}{ccc} E(A, N - B_\bullet) & \longrightarrow & E(A^+ \cup A^0, N - B_\bullet) \\ \downarrow & & \downarrow \\ E(A^0 \cup A^-, N - B_\bullet) & \longrightarrow & E(A^0, N - B_\bullet) \end{array} \tag{6.1}$$

of r-cubes in which the upper right and lower left cubes are cartesian, being made up of contractible spaces. The fibers of the vertical maps form a map of r-cubes

$$E(A^+, (N \backslash (A^0 \cup A^-) - B_\bullet) \longrightarrow E(A^+, (N \backslash A^0) - B_\bullet)$$

which according to B.1 will be a $(1-p+(n-p-2)+\sum_i(n-q_i-2))$-cartesian $(r+1)$-cube. It follows that diagram 6.1 itself is $(n - 2p - 1 + \sum_i (n - q_i - 2))$-cartesian. Now, 6.1 is mapped to another $(r+2)$-cube

$$\begin{array}{ccc} F(A, N - B_\bullet) & \longrightarrow & F(A^+ \cup A^0, N - B_\bullet) \\ \downarrow & & \downarrow \\ F(A^0 \cup A^-, N - B_\bullet) & \longrightarrow & F(A^0, N - B_\bullet). \end{array} \tag{6.2}$$

The cube 6.2 is ∞-cartesian, because for each subset S of $\{1, \ldots, r\}$ the square diagram

$$\begin{array}{ccc} F(A, N - B_S) & \longrightarrow & F(A^+ \cup A^0, N - B_S) \\ \downarrow & & \downarrow \\ F(A^0 \cup A^-, N - B_S) & \longrightarrow & F(A^0, N - B_S). \end{array}$$

is ∞-cartesian. It follows that the $(r+3)$-cube formed by 6.1 and 6.2 is $(n - 2p - 1 + \sum_i (n - q_i - 2))$-cartesian. We have to show that the $(r+1)$-cube

$$E(A, N - B_\bullet) \longrightarrow F(A, N - B_\bullet)$$

is $(n-2p-1+\sum_i(n-q_i-2))$-cartesian. The $(r+1)$-cubes

$$E(A^+\cup A^0, N-B_\bullet) \longrightarrow F(A^+\cup A^0, N-B_\bullet)$$
$$E(A^0\cup A^-, N-B_\bullet) \longrightarrow F(A^0\cup A^-, N-B_\bullet)$$

are ∞-cartesian, being made up of contractible spaces. Therefore it will be enough if the cube

$$E(A^0, N-B_\bullet) \longrightarrow F(A^0, N-B_\bullet)$$

is $(n-2p+\sum_i(n-q_i-2))$-cartesian. In fact, it is $(n-2(p-1)-1+\sum_i(n-q_i-2))$-cartesian by induction on p.

Again the case $p=0$ is easy.

7. Extensions to codimension ≤ 2

The hypothesis that the codimension is at least three can to a large extent be removed. This may seem a little odd, because it has been used in several essential ways; although perhaps it is fair to say that for most if not all of these purposes it would not have been hard to get by with just $n-p\geq 3$ and $n-q_i$ arbitrary.

In any case we now assume Conjecture B.1 as stated and discuss to what extent it then holds without the assumptions $n-p\geq 3$ and $n-q_i\geq 3$.

Recall first that A.1 gives B.1 in the case when $p=0$. By symmetry this also takes care of the case when $q_i=0$ for some i. (Use the equivalent D.1 in place of B.1.)

Now consider the case when $n-p\geq 3$ but $n-q_i$ is arbitrary. Let K be the number of i such that $n-q_i\leq 2$. We argue by induction on K. If B.1 holds for smaller values of K, then so does G.1. (In the argument that B.1 implies G.1, G.1 for $(n,p,q_1,\dots,q_r)$ depended on G.1 for $(n,p-1,q_1\dots,q_r)$ and B.1 for $(n,p,p,q_1,\dots,q_r)$.)

Let $N,A,B_1,\dots,B_r$ be as in B.1, and suppose without loss of generality that $n-q_r\leq 2$. We can assume that G.1 holds for $(n,p,q_1,\dots,q_r-1)$.

To show that $E(A,N-B_\bullet)$ is $(1-p+\sum_i(n-q_i-2))$-cartesian it will suffice (since $F(A,N-B_\bullet)$ is $(1-p+\sum_i(n-q_i-2))$-cartesian) to show that

$$E(A,N-B_\bullet) \longrightarrow F(A,N-B_\bullet)$$

is $(1-p+\sum_i(n-q_i-2))$-cartesian. Rewrite this as

$$\begin{array}{ccc} (E(A,(N\backslash B_{[r]})-B_\bullet) & \longrightarrow & F(A,(N\backslash B_{[r]})-B_\bullet) \\ \downarrow & & \downarrow \\ E(A,N-B_\bullet) & \longrightarrow & F(A,N-B_\bullet) \end{array}$$

where "$\bullet$" now refers to a subset of $\{1,\dots,r-1\}$ rather than of $\{1,\dots,r\}$. It will be enough if both the upper and the lower arrows here are $(2-p+\sum_i(n-q_i-2))$-cartesian. They are, because G.1 holds for both of them by induction and gives $n-2p-1+\sum_{i<r}(n-q_i-2)$, which is good enough since $n-p\geq 3$ and $n-q_r-2\leq 0$.

By symmetry, as soon as B.1 is known in the cases just treated, it is also known when $n-p=2$ and $n-q_i\geq 2$ for all i and $n-q_i\geq 3$ for some i.

It remains to discuss cases where $n-p\leq 2$ and $p\geq 1$ and $n-q_i\leq 2$ and $q_i\geq 1$ for all i. In most such cases $1-p+\sum_i(n-q_i-2)<$is negative and there is nothing to prove. The only exception is when $n=3$ and $p=1=q_i$ for all i. The question is whether $E(A,N-B_\bullet)$ is then 0-cartesian. I do not know the answer.

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