EXCISION ESTIMATES FOR SPACES OF HOMOTOPY EQUIVALENCES

Thomas G. Goodwillie

Brown University

Abstract. Motivated by applications to spaces of smooth embeddings, we obtain multirelative connectivity statements for spaces of self-homotopy equivalences of compact manifolds or finite Poincaré complexes.

1. Introduction

If $(N, \partial N)$ is either a compact manifold with boundary or a finite CW Poincaré pair, let $H(N)$ be the space of all homotopy equivalences from N to N fixing the boundary pointwise. (By a finite CW Poincaré pair we mean simply a finite CW pair satisfying relative Poincaré duality for twisted coefficients.) We obtain multirelative connectivity statements concerning the dependence of $H(N)$ on N. In a companion paper [5] we use these statements to deduce analogous statements about spaces of diffeomorphisms and related statements about spaces of smooth embeddings.

The latter statements play a key role in the calculus-of-functors method of studying homotopy types of spaces of smooth embeddings ([10], [11], [12], [13], [14], [7]). This method is akin to (and in fact predates) the calculus of homotopy functors developed in [2], [3], and [4]; in particular it is a machine that runs on multirelative connectivity estimates. The main purpose of this paper and of [5] is to provide high-grade fuel for that machine.

For a closed subset A of N, let $H(N\, rel\, A)$ be the subspace of $H(N)$ consisting of all homotopy equivalences from N to N fixed on the set $A \cup \partial N$. In practice A will always be a reasonable codimension-zero subobject of N. The boundary of A will be allowed to have two parts, one called $\partial_0 A$ that lies in the boundary of N and another called $\partial_1 A$ that extends into the interior. To be more precise:

Definition 1.1. Let $(N, \partial N)$ be an n-dimensional compact manifold with boundary [resp. a finite CW Poincaré pair of formal dimension n]. A subset A of N is a *codimension-zero subobject* if there are two compact manifold [resp. CW Poincaré] triads $(A; \partial_0 A, \partial_1 A)$ and $(N\backslash A; \partial_0(N\backslash A), \partial_1(N\backslash A))$, both of [formal] dimension n, such that N is the union of A and $N\backslash A$ along $\partial_1 A = \partial_1(N\backslash A)$ and ∂N is $\partial_0 A \cup \partial_0(N\backslash A)$.

The symbol $N\backslash A$ will always denote the "closed complement" of A in N as in the definition above.

The statements of the results will involve a number called the homotopy codimension of A in N, which in turn depends on the following definition:

Definition 1.2. The pair of spaces (K, L) has homotopy dimension $\leq p$ if it is a retract, up to homotopy relative to L, of a relative CW complex (K^*, L) with relative dimension $\leq p$. This means that there are maps $K \xrightarrow{i} K^* \xrightarrow{r} K$ and a homotopy from $r \circ i$ to 1_K, all fixed on L.

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Of course this implies that the relative cohomology of (K, L) (even with twisted coefficients) vanishes in degrees greater than p. If $p > 2$ and if (K, L) is CW then that implication is reversible according to a result of Wall [13].

Definition 1.3. The codimension-zero subobject A of N has homotopy codimension $\geq n - p$ if, in the notation of 1.1,

(a) the pair $(A, \partial_1 A)$ is $(n - p - 1)$-connected, and
(b) the pair $(A, \partial_0 A)$ has homotopy dimension $\leq p$.

For examples, think of a tubular or regular neighborhood of something p-dimensional in an n-manifold. If $p > 2$ then (a) implies (b).

There are maps (each of them a homeomorphism onto its image)

$$H(N\backslash A) \longrightarrow H(N\, rel\, A) \longrightarrow H(N) \tag{1.4}$$

The second of these is part of a fibration sequence

$$H(N\, rel\, A) \longrightarrow H(N) \longrightarrow N^{A\, rel\, \partial_0 A}. \tag{1.5}$$

Here the symbol $X^{K\, rel\, L}$ stands for the space of all continuous functions from K to X fixed on L, where "fixed" means coinciding with some given map $L \to X$ that is not made explicit. The statement that 1.5 is a fibration sequence means that the restriction map $H(N) \to N^{A rel \partial_0 A}$ is a fibration and that $H(N\, rel\, A)$ is one of its fibers. Nothing is being asserted about fibers over other components; in particular these other fibers might be empty.

Remark. The composed map in 1.4 really deserves a fibration sequence (up to homotopy) of its own

$$H(N\backslash A) \longrightarrow H(N) \longrightarrow P(A, N) \tag{1.6}$$

where $P(A, N)$ is a suitable space of "Poincaré embeddings" of A in N fixed on $A \cap \partial N$. For some remarks on the usefulness of this, see the introduction to [5].

Our results will refer to relative or multirelative versions of diagram 1.4. To get the idea, suppose that another codimension-zero subobject B of N is also given, disjoint from A. Then we have:

$$\begin{array}{ccccc}
H(N\backslash(A \cup B)) & \longrightarrow & H(N\backslash B\, rel\, A) & \longrightarrow & H(N\backslash B) \\
\downarrow & & \downarrow & & \downarrow \\
H(N\backslash A) & \longrightarrow & H(N\, rel\, A) & \longrightarrow & H(N)
\end{array}$$

Here the second row is 1.4 and the first row is like the second but with $N\backslash B$ in place of N. The general (multirelative) setting will involve several subobjects $B_1, \ldots, B_r$ of N, disjoint from A and from each other. For each subset S of $[r] = \{1, \ldots, r\}$, let B_S be the union of those B_i for which i belongs to S and let N_S be $N\backslash B_S$. For each S we have a diagram like 1.4, with N replaced by N_S. All in all we have a single diagram

$$H(N_\bullet\backslash A) \longrightarrow H(N_\bullet\, rel\, A) \longrightarrow H(N_\bullet) \tag{1.7}$$

of cubical diagrams. Each of the three symbols stands for an r-dimensional cubical diagram of spaces, and each arrow stands for an $(r + 1)$-dimensional cubical diagram of spaces. The Theorem and Corollary below will assert that certain cubical diagrams in this picture are k-cartesian, for certain numbers k that depend on the dimension of N and on the homotopy codimensions of A and B_i. (Saying that an r-dimensional cubical diagram is "k-cartesian" means essentially that its multirelative, or $(r + 1)$-ad, homotopy groups vanish in a range governed by k; for details see section 2 below.)

Here is the main result of this paper:

Theorem. *Let $r \geq 1$. Let $(N, \partial N)$ be either a compact manifold or a finite CW Poincaré pair. Let $(A, B_1, \dots, B_r)$ be disjointly embedded codimension-zero subobjects with respective homotopy codimensions $(n-p, n-q_1, \dots, n-q_r)$ all at least 3. Then the $(r+1)$-cubical diagram*

$$H(N_\bullet \backslash A) \to H(N_\bullet \, rel \, A)$$

is $(n - 2p - 3 + \sum_{i=1}^{r}(n - q_i - 2))$-cartesian.

Corollary (first form). *With the same hypothesis, the $(r+1)$-cubical diagram*

$$H(N_\bullet \backslash A) \to H(N_\bullet)$$

is $(-p + \sum_{i=1}^{r}(n - q_i - 2))$-cartesian.

For the rest of the paper we will often use the abbreviation

$$\Sigma = \sum_{i=1}^{r}(n - q_i - 2).$$

The corollary follows from the theorem (using the very easy Proposition 2.3 below) as soon as one sees that the diagram $H(N_\bullet \, rel \, A) \to H(N_\bullet)$ is $(-p + \Sigma)$-cartesian. That in turn follows, using the natural fibration sequence 1.5, from the standard fact proved in section 3):

Lemma 1.8. *With the same hypothesis, the diagram*

$$N_\bullet^{A \, rel \, \partial_0 A}$$

is $(1 - p + \Sigma)$ -cartesian.

If $k < 0$ then to say that some cubical diagram is k-cartesian is to say nothing; and of course for some choices of the numbers n, p, and q_i that is exactly what the Theorem or the Corollary says. In applications the general idea is to compensate for a large p by using a large r.

The Corollary can be given a more symmetrical appearance, because in it the set A plays the same role as the sets B_i:

Corollary (second form). *Let $(N, \partial N)$ be either a compact manifold or a finite CW Poincaré pair. If $B_1, \dots, B_r, r \geq 2$, are disjoint compact codimension-zero subobjects with respective homotopy codimensions $(n - q_1, \dots, n - q_r)$ all at least 3, then the $(r+1)$-cubical diagram $H(N_\bullet) = H(N \backslash B_\bullet)$ is $(2 - n + \Sigma)$-cartesian.*

The results stated above will be proved in the Poincaré case. The manifold versions follow immediately, given the subtle result [6] that every compact topological manifold is homotopy equivalent to a finite CW complex. Of course, for the applications to embeddings we are thinking mainly of smooth manifolds, where this subtlety does not arise. The finiteness should really be irrelevant; it comes up here only because of the old-fashioned way in which we make normal spherical fibrations in section 7.

The paper is organized as follows.

Section 2 explains the terms "k-cartesian" and "k-cocartesian".

Section 3 recalls from [3] some extensions of the Blakers-Massey triad connectivity theorem. These are basic tools for proving that cubical diagrams are k-cartesian or k-cocartesian. A simple consequence is Lemma 1.8. (More elaborate applications follow in sections 4, 5, and 6.)

Section 4 gives a quick proof of a weak form of the Corollary (the number is off by one) without using the Theorem.

Section 5 begins the proof of the Theorem by using the method of section 4 to reduce the problem to a stable one.

Section 6 solves the stable problem, using a duality principle.

Section 7 explains the duality principle.

Perhaps a few words should be said here about the logical relationship between this paper and [5].

The main business of [5] will be to prove diff analogues of the Theorem and Corollary stated above, that is, the statements but with spaces of diffeomorphisms substituted for spaces of homotopy equivalences. This will be done in a rather straightforward way, using block diffeomorphisms as an intermediate step between homotopy equivalences and diffeomorphisms. Thus the Theorem above will imply its block analogue using (unobstructed) surgery theory, and that in turn will imply the diff analogue using the multirelative disjunction lemma for pseudoisotopies [1].

On the other hand, there is a labor-saving device available. Anyone who chooses to skip sections 5 through 7 below can still see the same smooth results proved by a slightly less direct method. In fact, the weak Corollary proved in section 4 implies the analogous weak diff corollary, which in turn implies a weak diff theorem, which in turn implies the strong diff corollary, which in turn implies the strong diff theorem. This bootstrap argument does not seem to have an analogue in the setting of homotopy equivalences. In fact I do not know how to deduce the Theorem stated above from its Corollary

Remark 1.9. The results stated above are all concerned with spaces of homotopy equivalences from a space to itself. There is a mild generalizaton (suggested by Michael Weiss) that deals with homotopy equivalences from one space to another. In fact, let $\{N, A, B_i\}$ and $\{N', A', B'_i\}$ be two sets of data for the theorem and suppose given homeomorphisms $\partial N \to \partial N'$, $A \to A'$, and $B_i \to B'_i$ agreeing on the intersections. Then the Theorem and Corollary above are still true if $H(N_S)$, $H(N_S\, rel\, A)$, and $H(N_S \backslash A)$ are respectively replaced by $H(N_S, N'_S)$, the space of homotopy equivalences from N_S to N'_S fixed on ∂N_S; $H(N_S, N'_S\, rel\, A)$, the space of homotopy equivalences from N_S to N'_S fixed on $A \cup \partial N_S$; and $H(N_S \backslash A, N'_S \backslash A')$, the space of homotopy equivalences from $N_S \backslash A$ to ${N'}_S \backslash A'$ fixed on $\partial(N_S \backslash A)\}$.

One can even go a little further and suppose only that the maps $\partial N \to \partial N'$, $A \to A'$, $B_i \to {B'}_i$ are homotopy equivalences, agreeing on the intersections and giving homotopy equivalences $\partial_\varepsilon A \to \partial_\varepsilon A'$, $\partial_\varepsilon B_i \to \partial_\varepsilon {B'}_i$, and

$$(\partial N)\backslash(\partial_0 A \cup \partial_0 B_1 \cup \dots \cup \partial_0 B_r) \longrightarrow (\partial N')\backslash(\partial_0 A' \cup \partial_0 {B'}_1 \cup \dots \cup \partial_0 {B'}_r).$$

The proofs go through with no change at all.

Of course, these spaces of maps can sometimes be empty. If $H(N, N')$ is empty then we have cubical diagrams of empty spaces, and these are trivially ∞-cartesian. On the other hand, if $H(N_{[r]} \backslash A, {N'}_{[r]} \backslash A')$ is not empty then the choice of a point in it makes the space $H(N_S, {N'}_S)$ homotopy equivalent to $H(N_S)$ for all S; in this case the new results follow from the old ones. The genuinely new case is when the smallest space $H(N_{[r]} \backslash A, {N'}_{[r]} \backslash A')$ is not known in advance to be nonempty but the other spaces are.

2. Multirelative connectivity conventions

These are essentially as in section 1 of [3], where further details can be found.

A space is called k-connected, for $k \geq 0$, if it is path-connected and if its homotopy groups $\pi_i(X)$ are trivial for $1 \leq i \leq k$. Nonempty spaces are (-1)-connected, all spaces are k-connected if $k \leq -2$, and ∞-connected means k-connected for all k.

A pair (X, Y) or a map $Y \to X$ of spaces is called 0-connected if $\pi_0(Y)$ maps onto $\pi_0(X)$. It is called k-connected, for $k > 0$, if $\pi_0(Y)$ maps bijectively to $\pi_0(X)$ and (for all basepoints in Y) $\pi_i(Y)$ maps isomorphically to $\pi_i(X)$ for $1 \leq i < k$ and $\pi_k(Y)$ maps onto $\pi_k(X)$. Every map is k-connected if $k \leq -1$.

More succinctly, a map is k-connected if and only if all of its homotopy fibers are $(k-1)$-connected spaces.

The statement that a given square diagram of spaces

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ C & \longrightarrow & D \end{array} \tag{2.1}$$

is k-cartesian means, roughly, that its bi-relative (or "triad") homotopy groups vanish up to dimension $k+1$. It actually means that that statement holds for every possible choice of basepoint (in A); and it also means a bit more. The precise requirement is that the canonical map from A to the homotopy inverse limit of the diagram formed by the other three spaces (also known as the homotopy fiber product of B and C over D) is k-connected.

The generalization from squares to r-dimensional cubes is as follows:

By an r-cubical diagram, or r-cube, of spaces, we mean a functor from $P([r])^{op}$ to the category of spaces, where $P([r])^{op}$ is the poset of all subsets of $[r] = \{1, \ldots, r\}$. The ordering in $P([r])^{op}$ is by reverse inclusion. (This is the only departure from the conventions of [3], where an r-cube was a functor from the opposite poset $P([r])$. The change is inessential since $P([r])$ is isomorphic to $P([r])^{op}$.)

Such a diagram arises, for example, from any $(r+1)$-ad. An $(r+1)$-ad of spaces is a space X together with subspaces $\{X_j, j \in [r]\}$. An $(r+1)$-ad determines an r-cube $X_\bullet$ by letting X_ϕ be X and otherwise letting X_S be the intersection of those X_j for which j belongs to S.

An r-cubical diagram $X_\bullet$ is called k-cartesian if the canonical map

$$X_{[r]} \xrightarrow{a(X_\bullet)} h(X_\bullet)$$

associated with it is k-connected. The space $h(X_\bullet)$ is the homotopy inverse limit of the diagram (= functor) obtained by restricting X to the full subcategory of $P([r])^{op}$ having all objects except the initial object $[r]$.

The multirelative homotopy group (if $i > r$, or set if $i = r$) $\pi_i(X_\bullet)$ may be defined as π_{i-r} of a certain space associated with the cubical diagram $X_\bullet$. (This space was called the "total fiber" in [3].) This definition depends on a choice of basepoint in $X_{[r]}$, and it requires that $i \geq r$. The total fiber is the same as the homotopy fiber of the map $a(X_\bullet)$ over the basepoint in $h(X_\bullet)$. Therefore the statement that $X_\bullet$ is k-cartesian implies that $\pi_i(X_\bullet)$ is trivial for $r \leq i \leq k+r-1$; it implies this for every choice of basepoint in $X_{[r]}$. In fact, it says more, because it also implies (if $k \geq 0$) that $\pi_0(X_{[r]})$ maps onto $\pi_0(h(X_\bullet))$.

Of course a 1-cube is the same as a map of spaces and in this case "k-cartesian cube" means "k-connected map". A 0-cube is a space and in this case "k-cartesian cube" means "$(k-1)$-connected space" because $h(X_\bullet)$, the homotopy inverse limit of the empty diagram, is a point.

Suppose that $X_\bullet \to Y_\bullet$ is a map (= natural transformation) of r-cubes; such a map can be considered as an $(r+1)$-cube. The following is Proposition 1.6 of [3]:

Lemma 2.2. *If both $Y_\bullet$ and $X_\bullet \to Y_\bullet$ are k-cartesian, then $X_\bullet$ is k-cartesian. If $X_\bullet$ is k-cartesian and $Y_\bullet$ is $(k+1)$-cartesian, then $X_\bullet \to Y_\bullet$ is k-cartesian.*

Warning. 2.2 can yield conclusions about $X_\bullet$ or about $X_\bullet \to Y_\bullet$ but not about $Y_\bullet$. The expected statement would be that if $X_\bullet \to Y_\bullet$ is $(k-1)$-cartesian and $X_\bullet$ is k-cartesian then $Y_\bullet$ is k-cartesian. This is not true in general, because of difficulties with π_0. For example, if B is empty in the 2-cube 2.1 above then trivially both the 1-cube $A \to B$ and the 2-cube are ∞-cartesian although $C \to D$ may be any 1-cube at all.

The next result follows from 2.2; it is Proposition 1.8 of [3]:

Lemma 2.3. *If the maps $X_\bullet \to Y_\bullet$ and $Y_\bullet \to Z_\bullet$ of r-cubes are k-cartesian (as $(r+1)$-cubes), then the composed map $X_\bullet \to Z_\bullet$ is also k-cartesian.*

The following is Proposition 1.18 of [3]. The "if" part is only slightly less trivial than the "only if".

Lemma 2.4. *The map of r-cubes $X_\bullet \to Y_\bullet$ is k-cartesian if and only if for every choice of point in $Y_{[r]}$ the r-cube of homotopy fibers is k-cartesian.*

For example, the 2-cube 2.1 above is k-cartesian if and only if, for every point in B, the induced map from the homotopy fiber of $A \to B$ to the homotopy fiber of $C \to D$ is k-connected. By symmetry there is also an equivalent criterion using the homotopy fibers of $A \to C$ and $B \to D$.

Warning. In applying 2.4, all points of $Y_{[r]}$ must be checked. (Of course, checking one point from each path-component is enough, so if the space $Y_{[r]}$ happens to be path-connected then one point is enough.)

The following consequence of 2.2 will be useful; it is a special case of Proposition 1.20 of [3].

Lemma 2.5. *Given an $(r+2)$-cube of spaces displayed as a square of r-cubes:*

$$\begin{array}{ccc} A_\bullet & \longrightarrow & B_\bullet \\ \downarrow & & \downarrow \\ C_\bullet & \longrightarrow & D_\bullet \end{array}$$

suppose that $B_\bullet$ and $C_\bullet$ and the $(r+2)$-cube are all k-cartesian, and that $D_\bullet$ is $(k+1)$-cartesian. Then $A_\bullet$ is k-cartesian.

Remark 2.6. The statement that a pair (K, L) of CW complexes has homotopy dimension $\leq p$ implies (and is in fact equivalent to) the statement that, for every k-connected map $X \to Y$, the diagram

$$\begin{array}{ccc} X^K & \longrightarrow & X^L \\ \downarrow & & \downarrow \\ Y^K & \longrightarrow & Y^L \end{array}$$

of function spaces is $(k-p)$-cartesian. This in turn implies the more general statement that for every k-cartesian cubical diagram $X_\bullet$ of spaces the diagram

$$X_\bullet^K \longrightarrow X_\bullet^L$$

is $(k-p)$-cartesian, since the functors $()^K$ and $()^L$ commute with homotopy limits.

The dual notion of k-cocartesian cube will also play a role here. An n-cube $X_\bullet$ is called k-cocartesian if the canonical map

$$k(X_\bullet) \xrightarrow{b(X_\bullet)} X_\phi$$

associated with it is k-connected, where the space $k(X_\bullet)$ is the homotopy colimit of the diagram obtained by restricting X to the full subcategory of $P([r])^{op}$ having all objects except the final object ϕ. Again see section 2 of [3] for details.

Proposition 2.7. *In an ∞-cocartesian 2-cube*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

if $A \to B$ is 2-connected and $B \to D$ is k-connected then $A \to C$ is k-connected.

Proof. We have excision isomorphisms $H_i(A \to C) \to H_i(B \to D)$, for for homology with coefficients in any system on D. Therefore $H_i(A \to C) = 0$ for $i \leq k$, for every coefficient system on C that is pulled back from D. Since $C \to D$ is 2-connected, that means all systems on C. This homological connectivity implies that $A \to C$ is k-connected if it induces isomorphisms of fundamental groups. It does so if $k \geq 2$, for then the other three maps do so. If $k < 2$ then we can simply say that $A \to C$ is k-connected because $A \to D$ is k-connected and $C \to D$ is $(k+1)$-connected.

3. Review of the multirelative Blakers-Massey theorem and its dual

In one form the classical Blakers-Massey triad connectivity theorem says that an ∞-cocartesian square diagram

$$\begin{array}{ccc} X_{12} & \xrightarrow{f} & X_2 \\ {\scriptstyle g}\downarrow & & \downarrow \\ X_1 & \longrightarrow & X \end{array}$$

is $(a+b-1)$-cartesian if f is a-connected and g is b-connected. We recall from section 2 of [3] some related statements about higher-dimensional cubes. Call a cube $X_\bullet$ *strongly cocartesian* if each of its two-dimensional faces is ∞-cocartesian.

Theorem 3.1 (2.3 of [3]). *Suppose that the r-cubical diagram $X_\bullet$ is strongly cocartesian and that for each $i \in [r]$ the map $X_{[r]} \to X_{[r]-\{i\}}$ is k_i-connected. Then $X_\bullet$ is $(1 + \sum_{i=1}^{r}(k_i - 1))$-cartesian.*

For example, let $N_\bullet = N \backslash B_\bullet$ be defined by disjoint codimension-zero subobjects $B_1, \ldots, B_r$ of N as in section 1. Then $N_\bullet$ is strongly cocartesian because each two-dimensional face is a pushout square of cofibrations. Furthermore, if B_i has homotopy codimension $\geq n - q_i$ then the pair $(N_{[r]-\{i\}}, N_{[r]})$ is $(n - q_i - 1)$-connected. It follows that the diagram $N \backslash B_\bullet$ is $[1 + \sum_{i=1}^{r}(n - q_i - 2))$-cartesian.

Together 2.6 and 3.1 yield a statement that contains Lemma 1.8, and that will also be used repeatedly below:

Corollary 3.2. *Let $X_\bullet$ and the numbers k_i be as in 3.1, and suppose that (K, L) is a CW pair with homotopy dimension $\leq p$. Then the $(r+1)$-cubical diagram $X_\bullet^K \to X_\bullet^L$ is $(1 - p + \sum_{i=1}^{r}(k_i - 1)]$-cartesian. Thus for any map $L \to X_{[r]}$ the r-cubical diagram $X_\bullet^{K\, rel\, L}$ is $(1 - p + \sum_{i=1}^{r}(k_i - 1))$-cartesian.*

There is a more general statement than 3.1 that applies when the cube is not strongly cocartesian. It will not be needed until [5]:

Theorem 3.3 (2.5 of [3]). *Let $X_\bullet$ be an r-cube of spaces. If, for each nonempty subset S of $[r]$, there is given an integer $k(S)$ such that the cube formed by the spaces X_T, T containing the complement of S, is $k(S)$-cocartesian, and if $k(S) \leq k(T)$ whenever S is contained in T, then $X_\bullet$ is k-cartesian where k is the minimum of $1 - r + \Sigma_\alpha k(T_\alpha)$ over all partitions $\{T_\alpha\}$ of $[r]$ into disjoint nonempty sets.*

There are also dual statements for deducing that cubes are k-cocartesian. The dual of 3.1 will not be needed; the dual of 3.3 is:

Theorem 3.4 (2.6 of [G3]). *Let $X_\bullet$ be an r-cube of spaces. If, for each nonempty subset S of $[r]$, there is given an integer $k(S)$ such that the cube formed by the spaces X_T, T contained in S, is $k(S)$-cartesian, and if $k(S) \leq k(T)$ whenever S is contained in T, then $X_\bullet$ is k-cocartesian where k is the minimum of $r - 1 + \Sigma_\alpha k(T_\alpha)$ over all partitions $\{T_\alpha\}$ of $[r]$ into disjoint nonempty sets.*

4. A quick proof

The next result is a not quite sharp form of the Corollary. We include its proof here partly because it is relatively easy and partly because it introduces a construction that will reappear in section 5.

Let N and $(B_1, \dots, B_r)$ be as in the Corollary (second form). Write $[N_S, N_S]$ for the space of all continuous self-maps of $N_S = N \backslash B_S$ fixed on ∂N_S. Like the spaces $H(N_S)$, these form a cubical diagram. It can be regarded as a diagram of subspaces of $[N, N]$ and inclusion maps by identifying the elements of $[N_S, N_S]$ with those self-maps of N fixed on $\partial N \cup B_S$ that carry N_S into itself.

Lemma 4.1. *If N and $B_i (1 \le i \le r)$ are as in the Corollary, then the cubical diagrams $[N_\bullet, N_\bullet]$ and $H(N_\bullet)$ are $(1 - n + \Sigma)$-cartesian.*

To prove 4.1 we subdivide the cubical diagram $[N_\bullet, N_\bullet]$ into smaller cubes, as follows: If (T, S) is any pair of subsets of $[r] = \{1, \dots, r\}$ with $T \supset S$ (so that $N_S \supset N_T$), then let $[N_T, N_S]$ be the space of all continuous maps from N_T to N_S fixed on ∂N_T. Again we can identify $[N_T, N_S]$ with a subspace of $[N, N]$, namely the self-maps of N fixed on $\partial N \cup B_T$ and carrying N_T into N_S.

If (S', T') is another such pair with $S' \supset S$ and $T' \supset T$, then $[N_T, N_S] \supset [N_{T'}, N_{S'}]$. Thus we have a functor from the (opposite) poset of all such pairs (T, S) to spaces. Such a functor is a diagram in the shape of an r-dimensional cube subdivided into 2^r smaller cubes. For example, when $r = 2$ the diagram has the shape

$$\begin{array}{ccccc}
(\{1,2\},\{1,2\}) & \longrightarrow & (\{1,2\},\{1\}) & \longrightarrow & (\{1\},\{1\}) \\
\downarrow & & \downarrow & & \downarrow \\
(\{1,2\},\{2\}) & \longrightarrow & (\{1,2\},\phi) & \longrightarrow & (\{1\},\phi) \\
\downarrow & & \downarrow & & \downarrow \\
(\{2\},\{2\}) & \longrightarrow & (\{2\},\phi) & \longrightarrow & (\phi,\phi).
\end{array}$$

The large cube is $[N_\bullet, N_\bullet]$. In order for the large cube to be $(1 - n + \Sigma)$-cartesian it is enough (by repeated applications of 2.3) if each of the 2^r small cubes is $(1 - n + \Sigma)$-cartesian.

The small cubes may be enumerated like this: There is one for each subset U of $[r]$, namely $[N_{U \cup \bullet}, N_{U \cap \bullet}]$. Thus the extreme cases are $[N_{[r]}, N_\bullet]$ and $[N_\bullet, N_\phi]$. We will treat a small cube differently according to how many elements the set U has: $r, r - 1$, or fewer.

Claim A. *The cube $[N_{[r]}, N_\bullet]$ is $(1 - n + \Sigma)$-cartesian.*

Claim B. *If $U = [r] - \{j\}$ for some j, then the cube $[N_{U \cup \bullet}, N_{U \cap \bullet}]$ is $(2 - n + \Sigma)$-cartesian.*

Claim C. *In all other cases the cube $[N_{U \cup \bullet}, N_{U \cap \bullet}]$ is ∞-cartesian.*

Claim A follows from 3.2, taking $(K, L) = (N_{[r]}, \partial N_{[r]})$.

For Claim B, consider the fibration sequence

$$[N_{[r]}, N_S] \longrightarrow [N_{[r]-\{j\}}, N_S] \longrightarrow N_S^{B_j \; rel \; \partial_0 B_j},$$

where S is a subset of $U = [r] - \{j\}$. (As usual $\partial_0 B_j$ means $B_j \cap \partial N$.) Write $[N_{U \cup \bullet}, N_{U \cap \bullet}]$ as a map of $(r-1)$-cubes $[N_{[r]}, N_\bullet] \longrightarrow [N_{[r]-\{j\}}, N_\bullet]$, where "$\bullet$"now refers to a variable subset of U. We find that it is enough if the $(r-1)$-cube $N_\bullet^{B_j \; rel \; \partial_0 B_j}$ is k-cartesian where $k = 3 - n + \Sigma$. It is, by 3.2 again, this time with $(K, L) = (B_j, \partial_0 B_j)$ and $p = q_j$ (and "r"$= r - 1$).

To prove Claim C, choose two distinct elements i and j of $[r]-U$, and for a subset S of $[r]-\{i,j\}$ examine the square diagram

$$\begin{array}{ccc} [N_{S\cup\{i,j\}\cup U}, N_{S\cap U}] & \longrightarrow & [N_{S\cup\{i\}\cup U}, N_{S\cap U}] \\ \downarrow & & \downarrow \\ [N_{S\cup\{j\}\cup U}, N_{S\cap U}] & \longrightarrow & [N_{S\cup U}, N_{S\cap U}] \end{array}$$

This is a two-dimensional face of the cube $X_{U\cup\bullet}, N_{U\cap\bullet}$. It will be enough (by repeated applications of 2.2) if this square is ∞-cartesian for each S. It is, because it is part of a map of fibration sequences

$$\begin{array}{ccccc} [N_{S\cup\{i,j\}\cup U}, N_{S\cap U}] & \longrightarrow & [N_{S\cup\{i\}\cup U}, N_{S\cap U}] & \longrightarrow & N_{S\cap U}^{B_j\, rel\, \partial_0 B_j} \\ \downarrow & & \downarrow & & \downarrow \\ [N_{S\cup\{j\}\cup U}, N_{S\cap U}] & \longrightarrow & [N_{S\cup U}, N_{S\cap U}] & \longrightarrow & N_{S\cap U}^{B_j\, rel\, \partial_0 B_j} \end{array}$$

in which the third vertical map is an identity map.

This completes the proof of the statement in 4.1 about $[N_\bullet, N_\bullet]$. To deduce the other statement, first note that $H(N)$ is an open and closed subset of $[N,N]$, so that the cube $H(N)\cap[N_\bullet,N_\bullet]$ is $(1-n+\Sigma)$-cartesian as soon as the cube $[N_\bullet,N_\bullet]$ is. Then verify that $H(N_S)=H(N)\cap[N_S,N_S]$, in other words that a map from N_S to itself fixed on the boundary must be a homotopy equivalence if when extended by the identity to all of N it becomes a homotopy equivalence from N to N. This last follows by 2.7, since the inclusion of N_S into N is 2-connected. (This is the only place in the proof of 4.1 where the hypothesis $n-q_i\geq 3$ has been used.)

5. Reduction to a stable problem

We now begin the proof of the Theorem.

Standing hypothesis 5.1. $(N,\partial N)$ is a finite Poincaré complex, containing disjointly embedded codimension-zero subobjects $A, B_1, \dots, B_r, r\geq 1$, whose homotopy codimensions, respectively $n-p, n-q_1,\dots,n-q_r$, are all ≥ 3. Write M for $N\backslash A$ and M_S for $N_S\backslash A = M\backslash B_S$.

The theorem to be proved concerns the $(r+1)$-cubical diagram

$$H(M_\bullet)\to H(N_\bullet\, rel\, A). \tag{5.2}$$

Let $[M_S,N_S]$ be the space of all maps from M_S to N_S fixed on ∂M_S. We can think of this as a subspace of $[M,N]$ by extending maps to be the identity in B_S.

Remark 5.3. Diagram 5.2 results from the diagram

$$[M_\bullet, M_\bullet]\to[M_\bullet,N_\bullet]$$

of subspaces of $[M,N]$ (and inclusion maps) by intersecting them all with the open and closed subspace $H(N\, rel\, A)$. (This makes use of the assumptions $n-p\geq 3$ and $n-q_i\geq 3$ exactly as in the last paragraph of section 4.)

In proving the theorem it will be necessary to consider a slightly more general situation: For each S define a subspace of M_S by

$$M_S^0 = B_{[r]-S}\cup\partial M_S.$$

We have a map (inclusion) of r-cubes $M_\bullet^0 \xrightarrow{a} N_\bullet$ which is itself a strongly cocartesian $(r+1)$-cube.

Hypothesis 5.4. For some r-cube $Y_\bullet$ of spaces the inclusion map $M^0_\bullet \xrightarrow{a} N_\bullet$ is factored

$$M^0_\bullet \xrightarrow{b} Y_\bullet \xrightarrow{c} N_\bullet$$

the $(r+1)$-cube $c : Y_\bullet \longrightarrow N_\bullet$ is strongly cocartesian, and the map $c_{[r]} : Y_{[r]} \longrightarrow N_{[r]}$ is k-connected, $k \geq 2$.

The statement that $Y_\bullet \longrightarrow N_\bullet$ is strongly cocartesian implies that $Y_\bullet$ itself is strongly cocartesian. It also implies that the map $Y_S \xrightarrow{c_S} N_S$ is k-connected for all S as soon as it is so for $S = [r]$. It also implies that the map $Y_{[r]} \longrightarrow Y_{[r]-\{j\}}$ is $(n - q_j - 1)$-connected, by applying 2.7 to the square

$$\begin{array}{ccc} Y_{[r]} & \longrightarrow & N_{[r]} \\ \big\downarrow & & \big\downarrow \\ Y_{[r]-\{j\}} & \longrightarrow & N_{[r]-\{j\}} \end{array}$$

Clearly c induces a map of r-cubes $[M_\bullet, Y_\bullet] \longrightarrow [M_\bullet, N_\bullet]$, where $[M_S, Y_S]$ means maps from M_S to Y_S fixed (so as to coincide with b_S) on the boundary of M_S. The diagram in 5.3 is the special case in which Y is M (and b and c are inclusion maps and k is $n - p - 1$). In general the spaces Y_S are not required to satisfy Poincaré duality, and the maps $Y_S \longrightarrow N_S$ are not required to be injections.

If f denotes some chosen point in $[M_{[r]}, N_{[r]}]$ and also its image in $[M_S, N_S]$ for any S, then for each S the space

$$F_f(Y_S) = \mathrm{fiber}([M_S, Y_S] \longrightarrow [M_S, N_S])$$

is defined ("fiber" means homotopy fiber with respect to the point f). These spaces form an r-cube $F_f(Y_\bullet)$.

Hypothesis 5.5. The point f of $[M_{[r]}, N_{[r]}]$ belongs to the subspace $H(N_{[r]}\, rel\, A)$.

This implies that $M_{[r]} \xrightarrow{f} N_{[r]}$ is $(n - p - 1)$-connected, because it is the composition of a homotopy equivalence with the inclusion $M_{[r]} \longrightarrow N_{[r]}$. That in turn implies that more generally $M_S \xrightarrow{f} N_S$ is $(n - p - 1)$-connected.

In view of 5.3 and 2.4, the assertion to be proved is exactly this: Whenever 5.1 and 5.5 hold, then the cube $F_f(M_\bullet)$ is $(n - 2p - 3 + \Sigma)$-cartesian. We will prove that this is so by obtaining the more general statement:

Lemma 5.6. *If 5.1, 5.4, and 5.5 hold, and if $k \geq n-p-1$, then the cube $F_f(Y_\bullet)$ is $(k-p-2+\Sigma)$-cartesian.*

Actually 5.6 is not quite sharp; the proof will show that the number in it can be improved by one when $k > n - p - 1$. It seems likely that with some further work the same improvement could be made even when $k = n - p - 1$ (the case that is needed in the end). If so, this would lead to an improvement by one in the statement of the Theorem.

There are two steps in proving 5.6. The first is to move the problem into a stable range – stable with respect to a kind of fiberwise suspension. The key to this step is the next lemma.

Denote the mapping cylinder of $c = c_S : Y_S \longrightarrow N_S$ by CY_S and label the usual associated maps as follows:

$$\begin{array}{ccccc} Y_S & \xrightarrow{i} & CY_S & \xleftarrow{s} & N_S \\ & & {\scriptstyle\pi}\big\downarrow & & \\ & & N_S. & & \end{array}$$

Now form the diagram:

$$\begin{array}{ccccccc} M^0_S & \xrightarrow{b} & Y_S & \xrightarrow{i} & CY_S & & \\ & & {\scriptstyle c}\big\downarrow & & \big\downarrow & & \\ & & N_S & \longrightarrow & \Sigma Y_S & \longrightarrow & N_S. \end{array} \tag{5.7}$$

The space ΣY_s is defined so that the square is a pushout. The map from ΣY_S to N_S is defined so that the composed map from CY_S to N_S is the homotopy equivalence π and the composed map from N_S to N_S is the identity.

Since $Y_S \xrightarrow{c} N_S$ is k-connected, the map $\Sigma Y_S \to N_S$ must be $(k+1)$-connected. The reason is that the composition $CY_S \to \Sigma Y_S \to N_S$ is a homotopy equivalence while $CY_S \to \Sigma Y_S$, being obtained from c by pushout along the cofibration i, is k-connected.

A good way to think of this construction is that in the category of spaces over N_S the object Y_S has CY_S as its cone and ΣY_S as its (unreduced) suspension. The construction is natural with respect to S, and leads to a factorization $M^0_\bullet \to \Sigma Y_\bullet \to N_\bullet$ that satisfies 5.4 but with k replaced by $k+1$.

There is a diagram of cubical diagrams:

$$\begin{tikzcd}
{[M_\bullet, Y_\bullet]} \arrow[r] \arrow[d] & {[M_\bullet, CY_\bullet]} \arrow[d] & \\
{[M_\bullet, N_\bullet]} \arrow[r] & {[M_\bullet, \Sigma Y_\bullet]} \arrow[r] & {[M_\bullet, N_\bullet]}
\end{tikzcd}$$

Here each $[M_\bullet, ?_\bullet]$ is an r-cubical diagram of spaces, and each space $[M_S, ?_S]$ in that diagram is the space of all continuous maps from M_S to $?_S$ coinciding on ∂M_S with the map $M^0_S \to ?_S$ occurring in 5.7.

Lemma 5.8. *Given 5.1 and 5.4, the $(r+2)$-cubical diagram*

$$\begin{tikzcd}
{[M_\bullet, Y_\bullet]} \arrow[r] \arrow[d] & {[M_\bullet, CY_\bullet]} \arrow[d] \\
{[M_\bullet, N_\bullet]} \arrow[r] & {[M_\bullet, \Sigma Y_\bullet]}
\end{tikzcd}$$

is $(2k-1-n+\Sigma)$-cartesian.

To prove 5.8, think of the $(r+2)$-cube as an r-cube of squares, and subdivide it as in the proof of 4.1. Thus for each (T, S) there is the square

$$\begin{tikzcd}
{[M_T, Y_S]} \arrow[r] \arrow[d] & {[M_T, CY_S]} \arrow[d] \\
{[M_T, N_S]} \arrow[r] & {[M_T, \Sigma Y_S]}
\end{tikzcd}$$

For each subset U of $[r]$ there is an r-cube of squares which if displayed as a square of r-cubes looks like

$$\begin{tikzcd}
{[M_{U\cup\bullet}, Y_{U\cap\bullet}]} \arrow[r] \arrow[d] & {[M_{U\cup\bullet}, CY_{U\cup\bullet}]} \arrow[d] \\
{[M_{U\cup\bullet}, N_{N\cap\bullet}]} \arrow[r] & {[M_{U\cup\bullet}, \Sigma Y_{U\cap\bullet}]}
\end{tikzcd}$$

If for each U this $(r+2)$-cube of spaces is $(2k-1-n+\Sigma)$-cartesian, then the same will be true of the large cube.

Distinguish three cases as before:

Claim A. *The cube*

$$\begin{tikzcd}
{[M_{[r]}, Y_\bullet]} \arrow[r] \arrow[d] & {[M_{[r]}, CY_\bullet]} \arrow[d] \\
{[M_{[r]}, N_\bullet]} \arrow[r] & {[M_{[r]}, \Sigma Y_\bullet]}
\end{tikzcd}$$

is $(2k-1-n+\Sigma)$-cartesian.

Claim B. *If $U = [r] - \{j\}$ for some j then the cube*

$$\begin{array}{ccc} [M_{U\cup\bullet}, Y_{U\cap\bullet}] & \longrightarrow & [M_{U\cup\bullet}, CY_{U\cap\bullet}] \\ \big\downarrow & & \big\downarrow \\ [M_{U\cup\bullet}, N_{U\cap\bullet}] & \longrightarrow & [M_{U\cup\bullet}, \Sigma Y_{U\cap\bullet}] \end{array}$$

is $(2k - n + \Sigma)$-cartesian.

Claim C. *In all other cases the cube*

$$\begin{array}{ccc} [M_{U\cup\bullet}, Y_{U\cap\bullet}] & \longrightarrow & [M_{U\cup\bullet}, CY_{U\cap\bullet}] \\ \big\downarrow & & \big\downarrow \\ [M_{U\cup\bullet}, N_{U\cap\bullet}] & \longrightarrow & [M_{U\cup\bullet}, \Sigma Y_{U\cap\bullet}] \end{array}$$

is ∞-cartesian.

Claim A follows from 3.2, taking (K, L) to be $(M_{[r]}, \partial M_{[r]})$ and taking $X_\bullet$ to be the strongly cocartesian $(r + 2)$-cube

$$\begin{array}{ccc} Y_\bullet & \longrightarrow & CY_\bullet \\ \big\downarrow & & \big\downarrow \\ N_\bullet & \longrightarrow & \Sigma Y_\bullet \end{array}$$

Claim B follows from the statement that the cube

$$\begin{array}{ccc} Y_\bullet^{B_j\, rel\, \partial_0 B_j} & \longrightarrow & CY_\bullet^{B_j\, rel\, \partial_0 B_j} \\ \big\downarrow & & \big\downarrow \\ N_\bullet^{B_j\, rel\, \partial_0 B_j} & \longrightarrow & \Sigma Y_\bullet^{B_j\, rel\, \partial_0 B_j} \end{array}$$

(where $\bullet$ runs through subsets of U) is $(2k + 1 - n + \Sigma)$-cartesian, which in turn follows from 3.2.

For C recall that the r-cube $[M_{U\cup\bullet}, N_{U\cup\bullet}]$ was shown to be ∞-cartesian in the proof of 4.5, and observe that the same argument works for each of the other cubes $[M_{U\cup\bullet}\ , Y_{U\cup\bullet}], [M_{U\cup\bullet}, CY_{U\cap\bullet}]$, and $[M_{U\cup\bullet}, \Sigma Y_{U\cup\bullet}]$.

To begin proving 5.6, map $F_f(Y_S)$ to $\Omega F_f(\Sigma Y_S)$ by composing three maps:

$$\begin{array}{c} \text{fiber}([M_S, Y_S] \to [M_S, N_S]) \\ \sim\big\downarrow \\ \text{fiber}([M_S, Y_s] \to [M_S, CY_S]) \\ \big\downarrow \\ \text{fiber}([M_S, N_S] \to [M_S, \Sigma Y_S]) \\ \sim\big\downarrow \\ \Omega\, \text{fiber}([M_S, \Sigma Y_S] \to [M_S, N_S]) \end{array}$$

The first map, a homotopy equivalence, is induced by the section s of the homotopy equivalence $CY_S \xrightarrow{\pi} N_S$ together with a canonical homotopy between $s \circ \pi$ and the identity. The second map is induced by the square in 5.8. The third map, again a homotopy equivalence, exists because the composed map $N_S \to \Sigma Y_S \to N_S$ in 5.7 is the identity. All of this is natural with respect to S, and so there is a map $F_f(Y_\bullet) \to \Omega F_f(\Sigma Y_\bullet)$ of cubical diagrams.

The content of 5.8 is that this map is $(2k-1-n+\Sigma)$-cartesian. Since $2k-1-n \geq k-p-2$, the conclusion of 5.6 would follow if $\Omega F_f(\Sigma Y_\bullet)$ were known to be $(k-p-2+\Sigma)$-cartesian, for example if $F_f(\Sigma Y_\bullet)$ were known to be $(k+1-p-2+\Sigma)$-cartesian. We conclude that 5.6 is true for $Y_\bullet$ if it is true for $\Sigma Y_\bullet$.

6. Solution of the stable problem

The goal here is the following variant of 5.6:

Lemma 6.1. *Assume 5.1, 5.4, and 5.5, and assume also that the map of cubes $c : Y_\bullet \to N_\bullet$ has a section whose restriction to $M^0_\bullet$ is b. If the number k in 5.4 is large enough then the cube $F_f(Y_\bullet)$ is $(k-p-1+\Sigma)$-cartesian.*

This will imply 5.6, because fiberwise suspension of Y_S over N_S increases k by one and at the same time produces the required section. ("Large enough" will be a function of n , p , and the q_i.) There is a commutative diagram

$$\begin{array}{ccc} M^0_\bullet & \xrightarrow{b} & Y_\bullet \\ \downarrow & & \downarrow{\scriptstyle c} \\ M_\bullet & \xrightarrow[f]{} & N_\bullet \end{array}$$

Let $Y_S \to Z_S \to N_S$ be the canonical factorization of c_S as a homotopy equivalence followed by a (path) fibration. Let E_S be the fiber product of M_S and Z_S over N_S , so that there is a (commutative) square of cubes:

$$\begin{array}{ccc} E_\bullet & \longrightarrow & Z_\bullet \\ \downarrow & & \downarrow \\ M_\bullet & \longrightarrow & N_\bullet \end{array} \tag{6.2}$$

The given section of $Y_\bullet \to N_\bullet$ yields a section s_0 of $E_\bullet \to M_\bullet$, and the space $F_f(Y_S)$ is precisely the space of sections of the fibration $E_S \to M_S$ agreeing with s_0 on the boundary.

Using a duality principle $F_f(Y_S)$ can be identified, in a stable range, with another space

$$F_f(Y_S) \simeq \Omega^d Q\,\mathrm{cofiber}(T^\nu M_S \to T^\nu E_S).$$

Here Q means the homotopy colimit of $\Omega^i\Sigma^i$, ν is the normal spherical fibration of a Poincaré embedding of N in d-dimensional space, $T^\nu ?$ means Thom space of pullback of ν by some map $? \to N$ which is to be understood from the context, and "cofiber"means homotopy cofiber. A proof and a detailed statement of this duality principle will be given in section 7.

By 7.4 below there is a map of r-cubes

$$F_f(Y_\bullet) \to \Omega^d Q\,\mathrm{cofiber}(T^\nu M_\bullet \to T^\nu E_\bullet) \tag{6.3}$$

such that for each S the map $F_f(Y_S) \to \Omega^d Q\,\mathrm{cofiber}(T^\nu M_S \to T^\nu E_S)$ is $(2k-n-1)$-connected. It follows by 2.2 that the $(r+1)$-cube 6.3 is $(2k-n-1-r)$-cartesian. We have to prove that $F_f(Y_\bullet)$ is K-cartesian where

$$K = k-p-1+\Sigma.$$

If k is large enough then $2k-n-1-r \geq K$, and so it will suffice if $\Omega^d Q\,\mathrm{cofiber}(T^\nu M_\bullet \to T^\nu E_\bullet)$ is K-cartesian. The key to that is:

Claim 6.4. *The $(r+2)$-cube 6.2 is $(K+n+r+1)$-cocartesian.*

In fact, using 6.4 we find that the cube

$$\begin{array}{ccc} T^\nu E_\bullet & \longrightarrow & T^\nu Z_\bullet \\ \Big\downarrow & & \Big\downarrow \\ T^\nu M_\bullet & \longrightarrow & T^\nu N_\bullet \end{array}$$

is $(K+d+r+1)$-cocartesian. That makes the cube

$$\begin{array}{ccc} QT^\nu E_\bullet & \longrightarrow & QT^\nu Z_\bullet \\ \Big\downarrow & & \Big\downarrow \\ QT^\nu M_\bullet & \longrightarrow & QT^\nu N_\bullet \end{array}$$

$(K+d)$-cartesian (see 1.19 of [3]). On the other hand, $QT^\nu Z_\bullet \to QT^\nu N_\bullet$ is ∞-cartesian, since $Z_\bullet \to N_\bullet$ is ∞-cocartesian (it is strongly cocartesian, like $Y_\bullet \to N_\bullet$). Therefore $QT^\nu E_\bullet \to QT^\nu M_\bullet$ is $(K+d)$-cartesian and $\Omega^d Q\,\mathrm{cofiber}(T^\nu M_\bullet \to T^\nu E_\bullet) \approx \Omega^d\,\mathrm{fiber}(QT^\nu E_\bullet \to QT^\nu M_\bullet$ is K-cartesian.

Proof of 6.4. This uses 3.4. Index the spaces in 6.2 by subsets of a set $\{1,2,\ldots,r,\heartsuit,\spadesuit\}$ with $r+2$ elements, writing

$$X_S = N_S,\ X_{S\cup\{\heartsuit\}} = M_S,\ X_{S\cup\{\spadesuit\}} = Z_S,\ X_{S\cup\{\heartsuit,\spadesuit\}} = E_S,$$

for each subset S of $\{1,\ldots,r\}$. For each nonempty subset S^* of $\{1,2,\ldots,r,\heartsuit,\spadesuit\}$ we need a number $k(S^*)$ such that a certain subcube is $k(S^*)$-cartesian, namely the subcube consisting of those X_T with T contained in S^*. There are four cases.

If $S^* = S$ is contained in $\{1,\ldots,r\}$, then the cube is a subcube of $N_\bullet$. The latter is a strongly cocartesian cube in which the connectivities of the "initial edges "(that is, the maps $N_{[r]} \to N_{[r]-\{i\}}$) are the numbers $\{n-q_i-1\}_{1\le i\le r}$. The subcube is therefore strongly cocartesian, too, and its edges have connectivities $\{n-q_i-1\}_{i\in S}$. Therefore according to 3.1 it is $k(S)$-cartesian, where

$$k(S) = 1 + \Sigma_{i\in S}(n-q_i-2).$$

If $S^* = S\cup\{\heartsuit\}$ contains $\heartsuit$ but not $\spadesuit$, then we are dealing with a subcube of the strongly cocartesian cube $M_\bullet \to N_\bullet$, so 3.1 applies again. This time the edges have connectivities $\{n-q_i-1\}_{i\in S}$ and $n-p-1$, so the answer is

$$k(S\cup\{\heartsuit\}) = 1 + (n-p-2) + \Sigma_{i\in S}(n-q_i-2).$$

The case when S^* contains $\spadesuit$ but not $\heartsuit$ is similar, because the cube $Z_\bullet \to N_\bullet$ is also strongly cocartesian. This time the answer is

$$k(S\cup\{\spadesuit\}) = 1 + (k-1) + \Sigma_{i\in S}(n-q_i-2).$$

In the remaining case the answer is

$$k(S\cup\{\heartsuit\}\cup\{\spadesuit\}) = \infty$$

because each of the squares

$$\begin{array}{ccc} E_S & \longrightarrow & Z_S \\ \Big\downarrow & & \Big\downarrow \\ M_S & \longrightarrow & N_S \end{array}$$

is ∞-cartesian.

The sum of the numbers $k(T)$ over all sets T in a partition of $\{1,\ldots,r,\heartsuit,\spadesuit\}$ is ∞ if $\heartsuit$ and $\spadesuit$ are in the same part of the partition, and otherwise it is $c+(k-1)+(n-p-2)+\Sigma$, where c is the number of parts in the partition. The minimum value is $k+n-p-1+\Sigma$, and so the cube is $(K+n+r+1)$-cocartesian.

7. Duality

To complete the proof we have to explain what the map 6.3 is and why it is highly connected.

Let $(M, \partial_0 M, \partial_1 M)$ be a finite CW Poincaré triad of formal dimension n. Suppose that $E \to M$ is a k-connected fibration equipped with a section $s_0 : M \to E$. Write $G(E, M, \partial_0 M)$ for the space of all sections $M \to E$ agreeing with s_0 on $\partial_0 M$. Write $C(E, M, \partial_1 M)$ for the total cofiber (that is, the homotopy cofiber of homotopy cofibers) of the square diagram

$$\begin{array}{ccc} \partial_1 E & \longrightarrow & E \\ \uparrow & & \uparrow \\ \partial_1 M & \longrightarrow & M \end{array} \tag{7.1}$$

where $\partial_1 E$ is the fiber product of $E \to M \leftarrow \partial_1 M$, and where the vertical maps are given by s_0. We prove two lemmas. The first, 7.2, which applies only in the special case when M is a solid piece of n-space, identifies $G(E, M, \partial_0 M)$ with $\Omega^n QC(E, M, \partial_1 M)$ in a stable range. The second, 7.4, which applies in general, involves a twist by a spherical fibration. What is needed for 6.3 is the special case of 7.4 in which $\partial_1 M$ is empty.

Lemma 7.2. *Let $(M, \partial_0 M, \partial_1 M)$ be a compact PL n-dimensional manifold triad, and suppose that M is a subpolyhedron of $\mathbb{R}^n$. If $E \to M$ is a k-connected fibration equipped with a section $s_0 : M \to E$, then a certain canonical map*

$$D : G(E, M, \partial_0 M) \to \Omega^n QC(E, M, \partial_1 M)$$

is $(2k - n - 1)$-connected.

If ν is a spherical fibration over M, write $C^\nu(E, M, \partial_1 M)$ for the total cofiber of the square diagram

$$\begin{array}{ccc} T^\nu \partial_1 E & \longrightarrow & T^\nu E \\ \uparrow & & \uparrow \\ T^\nu \partial_1 M & \longrightarrow & T^\nu M. \end{array} \tag{7.3}$$

This is just like 7.1 except that each space has been replaced by the Thom space of the spherical fibration obtained by pulling back ν.

Lemma 7.4. *Let $(M, \partial_0 M, \partial_1 M)$ be a finite CW Poincaré triad of formal dimension n and let ν be a normal spherical fibration of rank $d - n$ for M. If $E \to M$ is a k-connected fibration equipped with a section $s_0 : M \to E$, then a certain canonical map*

$$D : G(E, M, \partial_0 M) \to \Omega^d QC^\nu(E, M, \partial_1 M)$$

is $(2k - n - 1)$-connected.

To be honest, in both lemmas the "map" D will really only be defined in a weak sense, as a chain of maps in which any wrong-way map is a weak homotopy equivalence. It will be clear that the construction has enough naturality to justify 6.3.

Before beginning the proofs, there is a remark to make: Let $\Sigma E \to M$ be the fiberwise unreduced suspension of the fibration $E \to M$, and let $\Omega\Sigma E \to M$ be its fiberwise loopspace, a fibration whose fiber over a point in M is $\Omega\Sigma$ of the fiber of $E \to M$. The construction of the map D below will be such that there is a homotopy-commutative diagram

$$\begin{array}{ccccc} G(E, M, \partial_0 M) & \xrightarrow{\alpha} & G(\Omega\Sigma E, M, \partial_0 M) & = & \Omega G(\Sigma E, M, \partial_0 M) \\ {\scriptstyle D_E}\downarrow & & & & \downarrow{\scriptstyle \Omega D_{\Sigma E}} \\ \Omega^d QC^\nu(E, M, \partial_1 M) & = & \Omega\Omega^d Q\Sigma C^\nu(E, M, \partial_1 M) & \xrightarrow[\beta]{} & \Omega\Omega^d QC^\nu(\Sigma E, M, \partial_1 M). \end{array}$$

The map β is a homotopy equivalence, induced by a homotopy equivalence from $\Sigma C^\nu(E, M, \partial_1 M)$ to $C^\nu(\Sigma E, M, \partial_1 M)$. The evident map $E \to \Omega\Sigma E$ is $(2k-1)$-connected on fibers because the fibers of $E \to M$ are $(k-1)$-connected, and so the induced map α of spaces of sections is $(2k-1-n)$-connected.

From this we draw the useful conclusion that in order for D to be $(2k-1-n)$-connected it will be enough if D is $(2k-c)$-connected for some number c depending on M but not on k or E. (There is a downward induction: If $D_{\Sigma E}$ is $(2(k+1)-(c+1))$-connected then $\Omega D_{\Sigma E}$ is $(2k-c)$-connected, making D_E $(2k-c)$-connected as long as $c \geq n+1$.)

A more sophisticated conclusion to draw from the same diagram would be that we should really be considering the (pre-)spectrum determined by the spaces $\{G(\Sigma^i E, M, \partial_0 M) | i \geq 0\}$. The statement should be that there is a stable homotopy equivalence between that spectrum and $\Omega^d \Sigma^\infty C^\nu(E, M, \partial_1 M)$, in other words that the "cohomology spectrum" of $(M, \partial_0 M)$ with "coefficients" in the "fiberwise suspension spectrum" of $E \to M$ is equivalent to the "homology spectrum" of $(M, \partial_1 M)$ with the coefficients suitably twisted by the normal bundle. We will not try to make this idea any more precise, but see section of [4] for a related discussion. (Some annoying details crop up here, for example if "fiberwise suspension" is taken in the unreduced sense, as it probably should be in order for fibrations to go to fibrations, then it is not precisely adjoint to "fiberwise loop", a point which makes the preceding remark a little less transparent than it ought to be.)

We now prove 7.4. We may assume that in E the set $s_0(M)$ is a fiberwise deformation retract of some neighborhood. This can be achieved by adding a "fiberwise whisker" to E; it does not alter the homotopy type of $G(E, M, \partial_0 M)$ or of $C(E, M, \partial_1 M)$; and it has the advantage that $C(E, M, \partial_1 M)$ may now be redefined (without changing its weak homotopy type) using true cofibers rather than homotopy cofibers. (The point of this is to make diagrams like 7.7 below strictly commutative.)

Define D as the adjoint of the composed map

$$S^n \wedge G(E, M, \partial_0 M) \to (M/\partial M) \wedge G(E, M, \partial_0 M) \to C(E, M, \partial_1 M) \to QC(E, M, \partial_1 M)$$

where the third arrow is the usual inclusion, the first is given by the collapse map

$$S^n = R^n \cup \{\infty\} \to M/\partial M,$$

and the second is induced by the evaluation map

$$M \times G(E, M, \partial_0 M) \to E.$$

(This evaluation map carries the subspace $\partial_1 M \times G(E, M, \partial_0 M)$ into $\partial_1 E$ and carries the subspaces $\partial_0 M \times G(E, M, \partial_0 M)$ and $M \times \{s_0\}$ into $s_0(M)$.)

We will prove that D is $(2k-c)$-connected for some number c by using a sort of Mayer-Vietoris argument to reduce to special cases.

The first step is to reduce to the case when $\partial_0 M$ is empty. To that end, choose a collar C for $\partial_0 M$ in M. Writing $(C, \partial_0 M) = ([0,2] \times \partial_0 M, \{0\} \times \partial_0 M)$, let P be the closed complement of $[0,1] \times \partial_0 M$ in M so that $M = P \cup C$ and $P \cap C = [1,2] \times \partial_0 M$. There is a commutative square

$$\begin{array}{ccc} G(E, M, \partial_0 M) & \longrightarrow & G(E, P, \phi) \\ \downarrow & & \downarrow \\ G(E, C, \partial_0 M) & \longrightarrow & G(E, P \cap C, \phi) \end{array} \tag{7.5}$$

of restriction maps. (We are now using "E" indiscriminately for the restrictions of the fibration-with-section E to various subsets of M.) The square is ∞-cartesian, because it is a pullback square of fibrations.

Four maps "D" make a map from the square 7.5 to another square

$$\begin{array}{ccc} \Omega^n QC(E,M,\partial_1 M) & \longrightarrow & \Omega^n QC(E,P,\partial P) \\ \downarrow & & \downarrow \\ \Omega^n QC(E,C,\partial_1 C) & \longrightarrow & \Omega^n QC(E,P\cap C,\partial(P\cap C)) \end{array} \tag{7.6}$$

where we have written $\partial_1 C$ for $(\{2\}\times\partial_0 M)\cup([0,2]\times\partial\partial_0 M)$. We claim that the square 7.6 is also ∞-cartesian. This will follow from the fact that the square

$$\begin{array}{ccc} C(E,M,\partial_1 M) & \longrightarrow & C(E,P,\partial P) \\ \downarrow & & \downarrow \\ C(E,C,\partial_1 C) & \longrightarrow & C(E,P\cap C,\partial(P\cap C)) \end{array} \tag{7.7}$$

which underlies it is ∞-cocartesian.

We briefly indicate how 7.7 is defined and why it is ∞-cocartesian. The idea is to substitute the pair $(M,(M\backslash P)\cup\partial M)$ for $(P,\partial P)$ in defining $C(E,P,\partial P)$, and likewise to substitute $(M,(M\backslash C)\cup\partial_1 M)$ for $(C,\partial_1 C)$ and $(M,(M\backslash P)\cup(M\backslash C)\cup\partial M)$ for $(P\cap C),\partial(P\cap C)$. The fact that the square is ∞-cocartesian then comes down to the fact that the square of inclusion maps

$$\begin{array}{ccc} \partial_1 M & \longrightarrow & (M\backslash P)\cup\partial M \\ \downarrow & & \downarrow \\ (M\backslash C)\cup\partial_1 M & \longrightarrow & (M\backslash P)\cup(M\backslash C)\cup\partial M \end{array} \tag{7.8}$$

is a pushout square of cofibrations and therefore ∞-cocartesian.

In both 7.5 and 7.6 the lower left space is ∞-connected. Therefore the statement "D is ($2k$ - constant)-connected" will be true for the map of upper left spaces if it is true for the other two maps.

Turning now to the case when $\partial_0 M$ is empty, the claim is that the statement "D is ($2k$ - constant)-connected" will be true for a given M if it is true for P,Q, and $P\cap Q$, provided $M=P\cup Q$ in a good way. Here "good" means that both in P and in Q the set $P\cap Q$ is a compact locally flat codimension-zero PL submanifold transverse to the boundary, and also that the closed complements $P\backslash P\cap Q$ and $Q\backslash P\cap Q$ are disjoint. Again the argument is that four maps "D" give a map from one ∞-cartesian square

$$\begin{array}{ccc} G(E,M,\phi) & \longrightarrow & G(E,P,\phi) \\ \downarrow & & \downarrow \\ G(E,Q,\phi) & \longrightarrow & G(E,P\cap Q,\phi) \end{array}$$

to another:

$$\begin{array}{ccc} \Omega^n QC(E,M,\partial M) & \longrightarrow & \Omega^n QC(E,P,\partial P) \\ \downarrow & & \downarrow \\ \Omega^n QC(E,Q,\partial Q) & \longrightarrow & \Omega^n QC(E,P\cap Q,\partial(P\cap Q)). \end{array}$$

The last square is ∞-cartesian because it is related to the pushout square of cofibrations

$$\begin{array}{ccc} \partial M & \longrightarrow & (M\backslash P)\cup\partial M \\ \downarrow & & \downarrow \\ (M\backslash Q)\cup\partial M & \longrightarrow & (M\backslash P)\cup(M\backslash Q)\cup\partial M \end{array}$$

as 7.6 is related to 7.8.

Using the claim, one can reduce the problem (say, by means of a handle decomposition of M) to the case when M is an n-disk or empty. In the case of an n-disk the map D is, up to homotopy equivalence, the usual map $E \longrightarrow \Omega^n Q \Sigma^n E$. Since the space E is (in this case) $(k-1)$-connected, the map is $(2k-1)$-connected.

Before beginning the proof of 7.4, we recall the idea [8] of normal spherical fibration in the form in which it will be needed here:

Any finite CW complex M admits a homotopy equivalence $h : T \longrightarrow M$ from a compact d-dimensional PL submanifold T of $\mathbb{R}^d$ for some large d. If M is a Poincaré complex of formal dimension n, then the homotopy fibers of $\partial T \longrightarrow T$ have the integral homology of the $(d-n-1)$-sphere. At the expense of stabilizing (replacing T by $I \times T$ and d by $d+1$) if necessary, the homotopy fibers can even be assumed to have the homotopy type of the sphere. The normal spherical fibration of M in d-space may be defined as the (path) fibration ν in the canonical factorization

$$\partial T \xrightarrow{\sim} S^\nu M \xrightarrow{\nu} M$$

of the restriction of h to ∂T. The Thom space $T^\nu M$ is the homotopy cofiber of ν. (Up to homotopy equivalence, then, $T^\nu M$ is $T/\partial T$.)

There is also the relative version: If $(M, \partial M)$ is a finite CW pair, then there is a homotopy equivalence $(T, \partial_0 T) \xrightarrow{h} (M, \partial M)$ where $(T, \partial_0 T, \partial_1 T)$ is a d-dimensional PL manifold triad embedded in R^d. If $(M, \partial M)$ is a Poincaré pair of formal dimension n, then, after stabilizing if necessary, there is an ∞-cartesian square

$$\begin{array}{ccc}
\partial_0 T \cap \partial_1 T & \longrightarrow & \partial_1 T \\
\downarrow & & \downarrow \\
\partial_0 T & \longrightarrow & T \\
\downarrow & & \downarrow \\
\partial M & \longrightarrow & M
\end{array}$$

providing compatible spherical fibrations for M and ∂M.

For clarity we just give the proof of 7.4 in the case when $\partial_1 M$ is empty; recall that this is the case that was needed in section 6.

Given a finite Poincaré duality pair $(M, \partial M)$, choose a homotopy equivalence $(T, \partial_0 T) \xrightarrow{h} (M, \partial M)$, where $(T, \partial_0 T, \partial_1 T)$ is a d-dimensional PL manifold triad embedded in R^d. By 7.2 there is a $(2k-d-1)$-connected map

$$G(h^* E, T, \partial_0 T) \xrightarrow{D} \Omega^d Q C(h^* E, T, \partial_1 T)$$

and this is all that is needed because there are homotopy equivalences

$$\begin{aligned}
G(E, M, \partial M) &\approx G(h^* E, T, \partial_0 T) \\
C(h^* E, T, \partial_1 T) &\approx \text{cofiber}(\text{cofiber}(\partial_1 T \to T) \to \text{cofiber}(\partial_1 h^* E \to h^* E)) \\
&\approx \text{cofiber}(T^\nu M \to T^\nu E) \\
&\approx C^\nu(E, M, \phi).
\end{aligned}$$

In the general case one chooses a homotopy equivalence $h : (T; \partial_0 T, \partial_1 T) \to (M; \partial_0 M, \partial_1 T)$, where $(T; \partial_0 T, \partial_1 T, \partial_2 T)$ is a d-dimensional PL manifold 4-ad embedded in $\mathbb{R}^d$, and then applies 7.2 to the triad $(T; \partial_0 T, \partial_1 T \cup \partial_2 T)$.

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