Fall 2009 MA 271 Stable Homotopy: More about fibrations

September 30, 2009

1. Further details about the π_* LES of a fibration:

a. In class I proved the surjectivity of $p_* : \pi_n(E, F, e_0) \to \pi_n(B, b_0)$ for $n > 0$ using the defining RLP of fibrations. For the injectivity, suppose that $p_*[\alpha_0] = p_*[\alpha_1]$, where α_0 and α_1 are maps $I^n \to E$ taking the boundary to F and all faces except $I^{n-1} \times 1$ to e_0. Let $\beta : I^n \times I \to B$ be a homotopy between $p \circ \alpha_0$ and $p \circ \alpha_1$ such that β_t takes the boundary to b_0 for all t. Define a lifting of β on all faces of $I^n \times I$ except $I^{n-1} \times 1 \times I$, by using α_0 on $I^n \times 0$, α_1 on $I^n \times 1$, and e_0 on the remainder.Then β has a lifting (defined on all of $I^n \times I$) that extends this one. It necessarily takes the remaining face into F and thus provides the required sort of homotopy between α_0 and α_1.

b. The resulting LES, obtained by substituting $\pi_n(B, p(e_0))$ for $\pi_n(E, F, e_0)$ in the LES of the pair, does not go as far as π_0, but this can be corrected. In fact, for $e_0 \in E$, $b_0 = p(e_0)$, and $F = p^{-1}(b_0)$ there are homomorphisms $\pi_n(B, b_0) \to \pi_{n-1}(F, e_0)$ $(n > 1)$ which, with the homomorphisms i_* and p_*, make a LES of groups ending with $\pi_1(B, b_0)$; for $b_0 \in B$ and $F = p^{-1}(b_0)$ there is an action of the group $\pi_1(B, b_0)$ on the set $\pi_0(F)$ such that the subgroup fixing the class of $e_0 \in F$ is the image of $\pi_1(E, e_0)$, and such that two elements of $\pi_0(F)$ are in the same orbit if and only if they go to the same element of $\pi_0(E)$, and such that the image of $\pi_0(F)$ in $\pi_0(E)$ is the preimage of the element of $\pi_0(B)$ represented by b_0.

2. Observation: When $p : Y \to B$ has the HLP for A then a map $A \to B$ must have a lifting along p if it is homotopic to a map that has such a lifting. Indeed, suppose that $g : A \times I \to B$ is a homotopy from g_0 to g_1 and suppose that $f_0 : A \to Y$ is such that $p \circ f_0 = g_0$. Then by the HLP there exists $f : A \times I \to Y$ such that $p \circ f = g$, and in particular there exists $f_1 : A \to Y$ such that $p \circ f_1 = g_1$.

2a. Relative form of same observation: Suppose that p has the relative HLP for a pair (A, C) and that a map $g : A \to B$ is given, plus a lifting $C \to Y$ of the restriction of g. We want a lifting of g relative to C, meaning a lifting that restricts to the given lifting on C. Such a lifting of g exists if g is homotopic, by a homotopy fixed on C, to a map that has a lifting relative to C.

Lemma: A map of spaces is both a fibration and a weak homotopy equivalence if and only if has the RLP with respect to the inclusion $j_n : S^{n-1} \to D^n$ for every $n \geq 0$.

Proof: First assume that $p : Y \to B$ has the RLP w.r.t. all j_n.

To see that it is a fibration, i.e. that it has the RLP with respect to the inclusion $D^n \times 0 \to D^n \times I$, factor the latter inclusion as follows:

$$D^n \times 0 \to (D^n \times 0) \cup (S^{n-1} \times I) \cup D^n \times 1) \to D^n \times I.$$

The second arrow is isomorphic to j_{n+1}. The first may be obtained from j_n by pushout. Therefore p has the RLP with respect to each of them, and therefore also to their composition.

To see that p is a weak equivalence, use the fibers of p. For each fiber $F = p^{-1}(b_0)$, the map $F \to *$ is obtained by pullback from p (using the map $b_0 : * \to B$), so that that map inherits the RLP w.r.t. j_n from p. In other words, every map $S^{n-1} \to F$ can be extended to D^n; the fibers of p are ∞-connected. Now it follows from the LES of the fibration that p induces isomorphisms of π_n for all n, including a bijection for π_0.

Now to prove the converse suppose that p is both a fibration and weak equivalence. Note that by the LES every fiber of p is ∞-connected. To verify the LLP for the inclusion j_n, let $f : S^{n-1} \to Y$ and $g : D^n \to B$ be maps such that $p \circ f = g \circ j_n$. Pulling back p by g, we get another fibration $g^*p : g^*Y \to D^n$, again with ∞-connected fibers. Thus g^*Y is ∞-connected. Rewriting liftings as sections of the pullback, we find that the problem is the following:

There is a fibration over D^n with ∞-connected total space. A section is defined on S^{n-1}. We want to extend it to a section defined on all of D^n.

Certainly we can extend it to a map $s : D^n \to g^*E$. Now apply observation (2a) above: the identity map $D^n \to D^n$ is homotopic (relative to S^{n-1}) to a map $(g^*p) \circ s$ that has a section which restricts as desired to S^{n-1}.