Spring 2010 MA 2110 Manifolds
Handout # 3
Paracompactness and Partitions of Unity

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These arguments are essentially the same as what is on pages 51-55 of Lee's book, maybe simpliifed a little.

1 Definitions

A collection of subsets S_α of the space X is called *locally finite* if every point $p \in X$ has a neighborhood N such that there are only finitely many α for which $N \cap S_\alpha$ is non-empty. It is easy to see that an open cover must be locally finite if for each α there are only finitely many α' such that $S_\alpha \cap S_{\alpha'}$ is nonempty.

The open cover V_α is a *refinement* of the open cover U_β if for every α there exists β such that $V_\alpha \subset U_\beta$.

X is called *paracompact* if every open cover of X has a locally finite refinement.

X is called *locally compact* if points have arbitrarily small compact neighborhoods: for every point $p \in X$, for every open set $U \subset X$ such that $p \in U$, there exist an open set V and a compact set K such that $p \in V \subset K \subset U$. (Some authors use a weaker condition: every point has a compact neighborhood. For Hausdorff spaces the two definitions are equivalent.) Of course, locally Euclidean implies locally compact.

The *support* of a continuous function $f : X \to \mathbb{R}$ is the closure of the set of all points $x \in X$ such that $f(x) \neq 0$.

A collection of functions f_i on X is called *locally finite* if the collection of supports is locally finite. If this is the case, then of course the pointwise sum $\Sigma_i f_i$ is well-defined and continuous. A continuous *partition of unity* on X is a locally finite collection of continuous real-valued functions $f_i \geq 0$ whose sum is the constant function 1. If X is a smooth manifold and the functions are smooth then the partition of unity is said to be smooth.

A partition of unity f_i is *subordinate to* the open cover U_i if for every i the support of f_i is contained in U_i.

We may also consider a more general notion in which the partition of unity and the cover are not required to have the same indexing sets. Say that the partition of unity g_j is *weakly subordinate to* the open cover U_i if for every j there is some i such the support of g_i is contained in U_i. In this case we can make a partition of unity subordinate to U_i as follows: for every j choose some $i = \phi(j)$ such that $V_j) \subset U_i$ and set $f_i = \Sigma_j g_j$, summing over all j such that $\phi(j) = i$.

2 The Paracompactness Result

We show that manifolds are paracompact. The proof applies to any space that is locally compact, Hausdorff, and second countable.

The key is to find a collection of open sets U_α and compact sets $K_\alpha \subset U_\alpha$ such that (1) the union of the K_α is X and (2) for each α there are only finitely many α' such that $U_\alpha \cap U_{\alpha'}$ is nonempty.

This implies paracompactness, for then given any open cover $\mathcal{O}$ of X we can find a locally finite refinement as follows: Call an open set *small* if it is contained in some member of $\mathcal{O}$. For every α, K_α can be covered by small open sets $V_{\alpha,j}$ contained in U_α, with j ranging over a finite set that depends on α. The $V_{\alpha,j}$ for all α and j constitute an open cover of X by small sets. It is a locally finite cover because for every α there are (by (2)) only finitely many pairs (α', j'), such that $V_{\alpha,j} \cap V_{\alpha',j'}$ might be nonempty.

Our U_α and K_α will be indexed by integers $\alpha > 0$. To make them, we first choose a sequence V_α of precompact open sets (*precompact* means having compact closure) such that $\bar{V}_\alpha \subset V_{\alpha+1}$ and such that the union is X. Extend to all integers α by writing $V_\alpha = \emptyset$ for $\alpha \leq 0$. Then take $U_\alpha = V_{\alpha+1} - \bar{V}_{\alpha-2}$ and $K_\alpha = \bar{V}_\alpha - V_{\alpha-1}$. Conditions (1) and (2) are easily verified.

To come up with the sequence V_α it suffices to cover X by a sequence $W_1 \subset W_2 \subset \dots$ of precompact open sets. The closure of any W_i is contained in some later $W_{i'}$, so there is some subsequence W_{i_α} that can serve as V_α.

To make such a sequence W_i it is enough to have any sequence of precompact open sets whose union is X; then take W_i to be the union of the first i sets. For that, we just take any countable base for the topology and use those members of the base which are precompact. These still form a base, because of the fact that X is locally compact and Hausdorff. In particular their union is X.

3 Existence of Partitions of Unity

We prove that there is a partition of unity subordinate to any given open cover $\mathcal{O}$ of the manifold X. The proof relies not on the paracompactness result but rather on its proof.

Clearly for every point $p \in X$ and every neighborhood of p there exists a continuous real-valued function $f_p \geq 0$ on X that is supported in that neighborhood and satisfies $f_p(p) > 0$. In the case of a smooth manifold f can be chosen to be smooth.

Now return to the K_α and U_α of the previous section, and again let *small* be defined in terms of the open cover $\mathcal{O}$. Fix α. For each point $p \in K_\alpha$ there is a function $f_p \geq 0$ with $f_p(p) > 0$ having its support in a small open set contained in U_α. There is a finite collection of $p_{\alpha,j}$ such that at each point in the compact set K_α there is some j such that $f_{p_{\alpha,j}} > 0$.

Now as α varies (and j varies through a finite number of values for each α) the functions $f_{p_{\alpha,j}}$ form a locally finite family. Let $f > 0$ be their sum. The functions $f_{p_{\alpha,j}}/f$ form a partition of unity subordinate to $\mathcal{O}$ in the weak sense.