Spring 2010
MA 2110, Introduction to Manifolds
Handout #1
C^∞ Maps

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Smooth manifolds are global geometric objects whose local structure is all about smooth mappings. The word *smooth* can be used in different ways, but in this course it will mean C^∞. For a more detailed account than the following, see pages 581-586 in Lee's book.

If $f : U \to \mathbb{R}^m$ is a mapping whose domain U is an open subset of $\mathbb{R}^n$ then we say that f is differentiable at a point $a \in U$ if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that the limit of $|f(x) - f(a) - L(x-a)|/|x-a|$ as $x \to a$ is zero. Differentiability at a implies continuity at a. The linear map L is unique if it exists, because for every vector $v \in \mathbb{R}^n$ the vector $L(v)$ can be described as a directional derivative, the limit of $(f(a+tv) - f(a))/t$ as $t \to 0$. L is called the derivative of f at a and denoted by $f'(a)$.

In particular, if f is differentiable at a then the (first-order) partial derivatives of f exist, and the matrix expression of $f'(a)$ with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ is the usual Jacobian matrix.

The chain rule holds in the sense that when g and f are composable and both $f'(a)$ and $g'(f(a))$ exist then $(g \circ f)'(a)$ exists and equals the composition $g'(f(a)) \circ f'(a)$. The proof is not hard.

If $f'(x)$ exists for every $x \in U$ then we say that f is differentiable. The function $f' : U \to Hom(\mathbb{R}^n, \mathbb{R}^m) \cong \mathbb{R}^{mn}$ may or not be differentiable. If it is, then its derivative at a point a is a linear map $\mathbb{R}^n \to Hom(\mathbb{R}^n, \mathbb{R}^m)$. This second derivative of f at a can then be interpreted as a bilinear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, and so on. We say that f is a C^r mapping if the kth order derivative at a exists and is continuous for all $1 \leq k \leq r$, and a C^∞ mapping if the kth order derivative at a exists for all $k \geq 1$.

The existence and continuity of first-order partial derivatives implies C^1. The existence and continuity of partial derivatives of order $\leq r$ implies C^r.

C^2 implies the equality of mixed second-order derivatives.