Spring 2010
MA 2110, Introduction to Manifolds
Handout #2
Submanifolds of $\mathbb{R}^n$

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Here is a discussion of the concept of m-dimensional smooth submanifold of $\mathbb{R}^n$. Later it will be superseded by the general concept of submanifold of an abstract manifold, but right now I want to get some ideas across by looking at this concrete case.

1 Curves in the plane

Let us start with the case $m = 1$, $n = 2$. I will give several definitions of *smooth curve in the plane* and show that they are all logically equivalent.

Let $C \subset \mathbb{R}^2$ be a subset and let $P = (a, b) \in C$ be a point. The following four conditions are equivalent, and if they hold for every $P \in C$ we say that C is a smooth curve in $\mathbb{R}^2$.

(1) For some open set $U \subset \mathbb{R}^2$ containing P there exists a diffeomorphism $\Phi : U \to \Phi(U)$ from U to some open subset of $\mathbb{R}^2$ such that $\Phi(C \cap U) = (\mathbb{R} \times 0) \cap \Phi(U)$ and $\Phi(P) = (0, 0)$.

(2) (Locally C can be given a regular parametrization.) For some open subset V of $\mathbb{R}$ there is a smooth map $\phi : V \to \mathbb{R}^2$ such that $\phi(V)$ is a neighborhood of P in C, $\phi(0) = P$, and the derivative $\phi'(0)$ is not zero.

(3) (Locally C is a graph.) For some open set $U \subset \mathbb{R}^2$ containing P, the set $U \cap C$ can be described either as the set of all pairs (x, y) with $y = f(x)$ and $x \in J$ for some smooth f defined in an open interval $J \in \mathbb{R}$, or as the set of all pairs (x, y) with $x = g(y)$ and $y \in J$ for some open interval $J \in \mathbb{R}$.

(4) (Locally C is a regular level set.) For some open set $U \subset \mathbb{R}^2$ containing P there exists a smooth map $\psi : U \to \mathbb{R}$ such that $\psi^{-1}(0) = U \cap C$ and $\psi'(P)$ is not zero.

We outline the proof:

(1) implies (2) trivially. Define ϕ by $\phi(u) = \Phi^{-1}(u, 0)$ (the domain V being the set of all u such that $(u, 0) \in \Phi(U)$).

(2) implies (3) using the Inverse Function Theorem in one dimension. Write $\phi(u) = (\phi_1(u), \phi_2(u))$. Either $\phi_1'(0)$ or $\phi_2'(0)$ is nonzero, say the former. Restricting to a smaller interval if necessary, we can assume that ϕ_1 has an inverse. Let f be $\phi_2 \circ \phi_1^{-1}$. If $\phi_2'(0) \neq 0$ then let g be $\phi_1 \circ \phi_2^{-1}$.

(3) easily implies (4). If C is described locally by $y = f(x)$ then let $\psi(x, y) = y - f(x)$; if it is described by $x = g(y)$ then let $\psi(x, y) = x - g(y)$.

(4) implies (1) using the Inverse Function Theorem in two dimensions. One of the partial derivatives of ψ at P is nonzero, say $\partial_2\psi$. Define $\Phi : U \to \mathbb{R}^2$ by $\Phi(x, y) = (x - a, \psi(x, y))$. The derivative of Φ at P is an invertible two by two matrix, so after restricting to a smaller open neighborhood of P the map Φ becomes a diffeomorphism to its image.

2 The general case

Now let $0 \leq m \leq n$. Let $M \subset \mathbb{R}^n$ be a subset and let $P \in M$ be a point. The following four conditions are equivalent, and if they hold for every $P \in M$ we say that M is a smooth m-dimensional manifold in $\mathbb{R}^n$.

(1) For some open set $U \subset \mathbb{R}^n$ containing P there exists a diffeomorphism $\Phi : U \to \Phi(U)$ from U to some open subset of $\mathbb{R}^n$ such that $\Phi(M \cap U) = (\mathbb{R}^m \times 0) \cap \Phi(U)$ and $\Phi(P) = 0$.

(2) (Regular parametrization) For some open subset V of $\mathbb{R}^m$ there is a smooth map $\phi : V \to \mathbb{R}^n$ such that $\phi(V)$ is a neighborhood of P in M, $\phi(0) = P$, and the $n \times m$ derivative matrix $\phi'(0)$ has rank m (the maximum possible).

(3) (Graph) For some open set $U \subset \mathbb{R}^n$ containing P, the set $U \cap M$ is related by some permutation of the n standard coordinates in $\mathbb{R}^n$ to the set of all pairs $(x, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ with $y = f(x)$ for some smooth f defined for x in some open set $W \subset \mathbb{R}^m$.

(4) (Regular level set) For some open set $U \subset \mathbb{R}^n$ containing P there exists a smooth map $\psi : U \to \mathbb{R}^{n-m}$ such that $\psi^{-1}(0) = U \cap M$ and the $(n - m) \times n$ derivative matrix $\psi'(P)$ has rank $n - m$ (the maximum possible).

The arguments are essentially the same as in the case $m = 1$, $n = 2$. Here are details for the two most interesting steps.

(2) implies (3) using the Inverse Function Theorem in m dimensions. After some permutation of coordinates we can assume that the first m rows of $\phi'(0)$ constitute an invertible $m \times m$ matrix. Write $\phi(u) = (\phi_1(u), \phi_2(u)) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, so that $\phi_1'(0)$ is invertible. Restricting to a smaller

domain if necessary, we can assume that ϕ_1 has an inverse. Let f be $\phi_2 \circ \phi_1^{-1}$.

(4) implies (1) using the Inverse Function Theorem in n dimensions. After composing with a permutation we can assume that the last $n-m$ columns of $\psi'(P)$ constitute an invertible $(n-m) \times (n-m)$ matrix. Define $\Phi : U \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ by $\Phi(x, y) = (x - a, \psi(x, y))$ where $P = (a, b)$. The derivative of Φ at P is an invertible $n \times n$ matrix, so after restricting to a smaller open neighborhood Φ becomes a diffeomorphism to its image.

The fact that (4) implies (3) is a version of the Implicit Function Theorem.

3 Tangent Spaces

To a smooth m-manifold $M \subset \mathbb{R}^n$ and a point $P \in M$ is associated an m-dimensional vector subspace of $\mathbb{R}^n$, the *tangent space* $T_P M$. We can describe it in four ways corresponding to (1) through (4) above.

If Φ is a diffeomorphism as in (1) then we let $T_P M$ be $\Phi'(P)^{-1}(\mathbb{R}^m \times 0)$. To see that this is well-defined, first note that it does not change if Φ is replaced by its restriction to a smaller open neighborhood of P. Then suppose that Φ_1 and Φ_2 are two diffeomorphims as in (1) both having the same domain. Writing $\Phi_1 = \Psi \circ \Phi_2$, we have $\Phi_1'(P) = \Psi'(0) \circ \Phi_2'(P)$. Since the diffeomorphism $\Psi : \Phi_2(U) \to \Phi_1(U)$ preserves (a neighborhood of 0 in) $\mathbb{R}^m \times 0$, the linear isomorphism $\Psi'(0) : \mathbb{R}^n \to \mathbb{R}^n$ preserves $\mathbb{R}^m \times 0$ and therefore $\Phi_1'(P)^{-1}(\mathbb{R}^m \times 0) = \Phi_2'(P)^{-1}(\Psi'(0)^{-1}(0)(\mathbb{R}^m \times 0)) = \Phi_2'(P)^{-1}(\mathbb{R}^m \times 0)$.

Given a parametrization ϕ of a neighborhood of P in M as in (2), $T_P M = \phi'(0)(\mathbb{R}^m)$.

If M is locally the graph of f [altered by a permutation of the coordinates in $\mathbb{R}^n$] and $P = (a, b) = (a, f(a))$ then $T_P M$ is the graph of the linear map $f'(a) : \mathbb{R}^m \to \mathbb{R}^{n-m}$ [similarly altered].

Given a map ψ as in (4) such that M is locally $\psi^{-1}(0)$, the space $T_P M$ is the kernel of $\psi'(P) : \mathbb{R}^n \to \mathbb{R}^{n-m}$.

4 Transverse Intersections

Let M_1 and M_2 be smooth submanifolds of $\mathbb{R}^n$ with dimensions m_1 and m_2. We say that M_1 and M_2 are *transverse* (or *intersect transversely*) *at* $P \in M_1 \cap M_2$ if $T_P M_1 + T_P M_2 = \mathbb{R}^n$, or equivalently if the intersection $T_P M_1 \cap T_P M_2$ has vector space dimension $m_1 + m_2 - n$. Notice that if $m_1 + m_2 \geq n$ then this is the lowest possible dimension for the intersection of vector spaces of dimensions m_1 and m_2 in an n-dimensional vector space. We say simply that M_1 and M_2 are *transverse* if they are transverse at every point of intersection. If $m_1 + m_2 < n$ then M_1 and M_2 can never be transverse at a point, so the only way they can be transverse is by having empty intersection.

We show that if M_1 and M_2 are transverse and $m_1+m_2 \geq n$ then $M_1 \cap M_2$ is a smooth submanifold of $\mathbb{R}^n$ with dimension m_1+m_2-n, and that $T_P(M_1 \cap M_2) = T_P M_1 \cap T_P M_2$ for every P. For this we use the regular level set point of view: For a suitable open neighborhood U of $P \in M_1 \cap M_2$ choose $\psi_1 : U \to \mathbb{R}^{n-m_1}$ such that $\psi_1^{-1}(0) = U \cap M_1$ and $\psi_2 : U \to \mathbb{R}^{n-m_2}$ such that $\psi_2^{-1}(0) = U \cap M_2$, both with derivatives of maximal rank, and observe that because of the transversality the combined map

$$\psi = (\psi_1, \psi_2) : U \to \mathbb{R}^{n-m_1} \times \mathbb{R}^{n-m_2} = \mathbb{R}^{n-(m_1+m_2-n)}$$

also has derivative of maximal rank: the kernel of

$$\psi'(P) : \mathbb{R}^n \to \mathbb{R}^{n-(m_1+m_2-n)}$$

is the intersection of $ker(\psi_1'(P)) = T_P M_1$ and $ker(\psi_2'(P)) = T_P M_2$.