REVIEW OF SOME BASIC POINT-SET TOPOLOGY

SPACES

A space consists of a set \( X \) called the point set and a set of subsets of \( X \) called the open sets. There are three axioms:

1. The union of any set of open sets is open
2. The intersection of two open sets is open
3. \( X \) itself is open

Since the union of the empty set of sets is empty, Axiom 1 implies that the empty set is open. Axioms 2 and 3 together are equivalent to the axiom

\[' \text{2'. The intersection of any finite set of open sets is open,} \]

if we observe the convention that the intersection of the empty set of subsets of \( X \) is \( X \).

The set of all open sets is sometimes called the topology; thus a space consists of a set and a topology for that set. In practice one often uses the same name for the point set and for the space.

A set \( \mathcal{B} \) of open sets is called a basis for the topology if every open set is the union of some set of elements of \( \mathcal{B} \). Of course, the topology is determined by the basis.

Even if \( \mathcal{B} \) is not a basis for any topology, there is still a smallest topology for which the elements of \( \mathcal{B} \) are open. The open sets are those subsets of \( X \) that may be expressed as the union of sets each of which is the intersection of finitely many sets belonging to \( \mathcal{B} \). (One can check that this really is always a topology.) \( \mathcal{B} \) is then called a subbasis for that topology. It is a basis if the intersection of two elements of \( \mathcal{B} \) is always a union of elements of \( \mathcal{B} \) and if \( X \) itself is a union of elements of \( \mathcal{B} \).

One important way to make a topology is to use a metric, a function assigning a distance \( d(x, y) \geq 0 \) to each pair of points, subject to axioms saying that \( d(y, x) = d(x, y) \) and \( d(x, z) \leq d(x, y) + d(y, z) \) and \( d(x, y) = 0 \) iff \( x = y \). A set \( S \) is called open if for every \( x \in S \) there is some \( r > 0 \) such that \( S \) contains the ball \( B_r(x) \) consisting of all points \( y \) such that \( d(x, y) < r \). The balls are then open, and in fact the set of all balls is a basis.

The set \( \mathbb{R} \) of real numbers, with the usual metric topology, is a key example. The open intervals constitute a basis.

A subset \( N \subset X \) of a space is called a neighborhood of a point \( x \in X \) if \( N \) contains an open set that contains \( X \). Thus \( x \) belongs to every neighborhood of \( X \). A set \( S \subset X \) is open if and only if \( S \) is a neighborhood of every point of \( S \).

The set of all points \( x \) such that \( S \) is a neighborhood of \( x \) is called the interior of \( X \). Thus the interior of \( S \) is a subset of \( S \) and is always open. Furthermore, every open set contained in \( S \) is
contained in the interior of $S$, so we can say that the interior of $S$ is the largest open set contained in $S$. It is also the union of all the open sets contained in $S$.

Given a subset $S \subset X$, a point of $X$ is called a limit point of $S$ if every neighborhood of $x$ has nonempty intersection with $S$. (The term is sometimes used more narrowly, to mean that every neighborhood contains some point of $S$ different from $x$. To be a limit point in the broader sense is to be either a limit point in the narrow sense or a point of $S$.) $S$ is called closed if it contains all of its limit points. The closure of $S$ is the set of limit points of $S$.

Clearly $x$ is a limit point of $S$ if and only if $x$ has no neighborhood contained in the complement $X - S$. Thus $S$ is closed if and only if every point in its complement has a neighborhood inside the complement, if and only if the complement is open. The closure of $S$ is the complement of the interior of the complement of $S$. It is also the smallest closed set containing $S$, and the intersection of all the closed sets containing $S$.

(There is in general no smallest open set containing $S$, and no largest closed set contained in $S$.)

A given set $X$ has a largest topology, namely the one in which all sets are open. This is called the discrete topology, and $X$ is then called a discrete space. At the other extreme, the indiscrete topology has no open sets other than $X$ and $\emptyset$.

I am calling one topology larger than another when it has more open sets. One may also say that the one topology is finer and the other is coarser. (Probably it is not a good idea to say, as some do, that one topology is stronger or weaker than another, because historically these terms have not been used consistently. Presumably, those people who think that a finer topology is stronger are seeing the topology as something that pulls the points apart, while those who think that a coarser topology is stronger are seeing the topology as something that pulls the points together.)

**CONTINUITY**

If $X$ and $Y$ are (the point sets of) two spaces, then a map $f : X \to Y$ is continuous if for every open set $U \subset Y$ the preimage $f^{-1}(U)$ is an open set of $X$ (equivalently, if for every closed set $C \subset Y$ the preimage $f^{-1}(C)$ is a closed set of $X$, equivalently if for every subset $S \subset X$ the image of the closure of $S$ is contained in the closure of the image of $S$).

The composition of two continuous maps is always continuous. The identity map $1_X : X \to X$ is always continuous. Thus we have a category.

A homeomorphism between spaces $X$ and $Y$ is an isomorphism in this category, in other words a map $f : X \to Y$ such that $f$ is continuous and has a continuous inverse $Y \to X$, in other words a bijection such a set $U \subset X$ is open in $X$ if and only if $f(U)$ is open in $Y$.

**SUBSPACES**

If $X$ is a space and $A$ is a subset then $A$ becomes a space by giving it the smallest topology such that the inclusion $A \to X$ is continuous, namely by calling a subset $S$ of $A$ open if and only if it is the intersection of $A$ with some open subset of $X$. This is called the subspace topology or the
relative topology. "Subspace of $X$" always means subset of $X$ with the topology determined in this way by the topology of $X$.

Of course, when both $X$ and $A$ are being considered as spaces then in speaking of a subset of $A$ we have to be clear what we mean when we say that it is open, or closed.

By a slight generalization, if $f : Y \to X$ is any map of sets then a topology on $X$ determines a topology on $Y$, the smallest such that $f$ is continuous, by calling a subset of $Y$ open if and only if it is the preimage of some open subset of $X$. If $f$ is injective, then this makes $Y$ homeomorphic to the subspace $f(Y) \subset X$. If not, then it makes $Y$ an odd sort of space, in which some points are not closed.

Note that when $Y$ inherits its topology from $X$ in this way then, for any space $Z$ and any map $g : Z \to Y$, $g$ is continuous if and only if the composed map $f \circ g : Z \to X$ is continuous. To express this more informally in the most important case, a map from $Z$ to a subspace of $X$ is continuous if and only if it is continuous as a map to $X$.

QUOTIENT SPACES

By symmetry with the definitions above, if $f : X \to Y$ is any map of sets then a topology on $X$ determines a topology on $Y$, the largest such that $f$ is continuous, by calling a subset of $Y$ open if and only if its preimage is open in $X$.

Here the most important case is that in which $f$ is surjective. Up to homeomorphism this means that $Y$ can be made by specifying an equivalence relation on the set $X$ and letting the set $Y$ be the set of equivalence classes, and calling a subset of $Y$ open if and only if the union of that set of subsets of $X$ is open in $X$. $Y$ is then called a quotient space of $X$.

Note that when $Y$ inherits its topology from $X$ in this way then, for any space $Z$ and any map $h : Y \to Z$, $h$ is continuous if and only if the composed map $h \circ f : X \to Z$ is continuous. To express this more informally in the most important case, making a continuous map from a quotient space of $X$ to $Z$ is the same as making a continuous map from $X$ to $Z$ that yields a well-defined map of sets on the quotient.

Given a continuous surjection $f : X \to Y$, one sometimes wants to know whether it is a quotient map, that is, whether $f^{-1}(B) \subset X$ closed implies $B \subset Y$ closed. A continuous map $f : X \to Y$ is called \textit{closed} if for every closed subset $A \subset X$ the set $f(A) \subset Y$ is closed. A closed surjection is always a quotient map because for $B \subset Y$ $f^{-1}(B) \subset X$ closed implies $B = f(f^{-1}(B)) \subset Y$ closed.

CONNECTEDNESS

Call a space $X$ \textit{disconnected} if it is the union of two disjoint nonempty open sets (equivalently if it is the union of two disjoint nonempty closed sets, equivalently if it has a proper nonempty set that is both open and closed). Call $X$ \textit{connected} if it is not disconnected and not empty.

Call a subset of a space connected if, as a subspace, it is connected.
A point is connected.

If \( f : X \rightarrow Y \) is continuous and surjective, and if \( X \) is connected, then \( Y \) is connected. (Proof: Suppose for contradiction that \( Y = A \cup B \) with \( A \cap B = \emptyset \) and both \( A \) and \( B \) open and nonempty. Then \( X = f^{-1}(A) \cup f^{-1}(B) \) with \( f^{-1}(A) \cap f^{-1}(B) = \emptyset \) and both \( f^{-1}(A) \) and \( f^{-1}(B) \) open and nonempty.)

The union of two connected sets in a space is connected if the intersection is nonempty. (Proof: Suppose that \( X \cap Y \) has a point \( p \) in it and that \( X \) and \( Y \) are connected. If \( X \cup Y \) is the union of disjoint sets \( A \) and \( B \), both open in \( A \cup B \), then \( p \) belongs to \( A \) or \( B \), say \( A \). \( A \cap X \) is open and closed in \( X \) and nonempty, therefore \( A \cap X = X \). Likewise \( A \cap Y = Y \). Thus \( A = X \cup Y \) and \( B = \emptyset \).

In any space \( X \), define a relation between points by calling two points related if there exists a connected subset of \( X \) containing both of them. This is an equivalence relation. The equivalence classes are themselves connected. (Proof: If \( C \) is such a class, and if \( C = A \cup B \) with \( A \) and \( B \) disjoint, nonempty, and open in \( C \), then choose points \( a \in A \) and \( b \in B \). There is a connected set \( K \) in \( X \) containing both \( a \) and \( b \). \( K \) is contained in \( C \). \( K \) is the union of disjoint nonempty open sets \( A \cap K \) and \( B \cap K \), contradiction.) They are called the components or connected components of \( X \). Every connected subset of \( X \) is contained in a (unique) component, and the components are the largest connected subsets of \( X \).

The closure of a connected set is connected. (Proof: Assume that the closure of \( S \) is the union of two disjoint nonempty closed sets \( A \) and \( B \). If \( S \) is connected then one of the sets \( A \cap S \) and \( B \cap S \) must be \( S \), say the former. Then \( A \) contains \( S \). Since \( A \) is closed in the closure of \( S \), it must equal the closure.)

A closed interval in the number line is connected. This is a consequence of the completeness of the line: Suppose that in the interval \([a, b]\) the set \( S \) is open and closed and contains the point \( a \). Let \( T \) be the set of all \( x \) in \([a, b]\) such that \([a, x]\) is contained in \( S \). Let \( c \) be the least upper bound of \( T \). \( c \) belongs to \( S \) because \( c \) is in the closure of \( T \) and \( T \) is contained in \( S \) and \( S \) is closed. Since \( S \) is a neighborhood of \( a \), \( c > a \). If \( c < b \) then, again since \( S \) is open, \( S \) contains some interval \([d, e]\) with \( a < d < c < e < b \). There exists \( f \) in \( T \) between \( d \) and \( c \). Thus \( S \) contains \([a, f] \cup [d, e]\) = \([a, e]\), so that \( e \) is in \( T \) and \( c \) is not an upper bound for \( T \) after all. This contradiction shows that \( c = b \). Thus \( S \) is all of \([a, b]\).

A path in \( X \), from a point \( p \) to a point \( q \), is a continuous map \( \alpha : [0, 1] \rightarrow X \) such that \( \alpha(p) = 0 \) and \( \alpha(1) = q \). A nonempty space is called path-connected if there is a path from \( p \) to \( q \) for every \( p \) and \( q \). Because \([0, 1]\) is connected, every path-connected space is connected.

There is an equivalence relation on the point set of \( X \) defined by saying that \( p \) is equivalent to \( q \) if there is a path from \( p \) to \( q \) in \( X \). The equivalence classes are called path components, and are the maximal path-connected subspaces of \( X \).

Convex subsets of \( \mathbb{R} \) are path-connected and therefore connected. In addition to the closed intervals and the whole line \( \mathbb{R} \), these include the various other (open and half-open) intervals \((a, b)\), \([a, b)\), \((a, b]\) and half-lines \([a, \infty)\) and \((\infty, b]\) and the points. These are the only connected sets in \( \mathbb{R} \),
Metric spaces are Hausdorff. Subspaces of Hausdorff spaces are Hausdorff. Products of Hausdorff closed.

The identity map (as continuous inclusion followed by continuous projection. 

One can also make the product of more than two spaces, even infinitely many.

If $A \subset X$ and $B \subset Y$, then the set $A \times B$ appears to have two topologies, as a subspace of $X \times Y$ and as a product of two subspaces. These are in fact equal. We can see this as follows. Temporarily write $(A \times B)_s$ for the subspace of the product and $(A \times B)_p$ for the product of the subspaces.

The identity map $(A \times B)_s \to (A \times B)_p$ is continuous because the projections $(A \times B)_s \to A$ and $(A \times B)_s \to B$ are continuous, because the composed maps $(A \times B)_s \to A \to X$ and $(A \times B)_s \to B \to Y$ are continuous, because they can be factored $(A \times B)_s \to X \times Y \to A$ and $(A \times B)_s \to X \times Y \to B$ as continuous inclusion followed by continuous projection.

The identity map $(A \times B)_p \to (A \times B)_s$ is continuous because the inclusion $(A \times B)_p \to X \times Y$ is continuous, because the composed maps $(A \times B)_p \to X$ and $(A \times B)_p \to Y$ are continuous, because they can be factored $(A \times B)_p \to A \to X$ and $(A \times B)_p \to B \to Y$ as continuous projection followed by continuous inclusion.

HAUSDORFF SPACES

The space $X$ is a Hausdorff space if points can be separated by open sets, i.e. if for distinct points $p$ and $q$ in there always exist disjoint open sets $U$ and $V$ in $X$ such that $p \in U$ and $q \in V$.

This is equivalent to saying that in the product space $X \times X$ the diagonal set $\{(x,x)|x \in X\}$ is closed.

Metric spaces are Hausdorff. Subspaces of Hausdorff spaces are Hausdorff. Products of Hausdorff
spaces are Hausdorff.

COMPACTNESS

A space \( X \) is compact if every open cover has a finite subcover. This means that whenever a collection of open sets \( \{ V_\alpha \}_{\alpha \in A} \) is such that \( X \) is the union of \( V_\alpha \), then there exist \( \alpha_1, \ldots, \alpha_N \) (for some \( N \)) such that \( X = V_{\alpha_1} \cup \ldots \cup V_{\alpha_N} \). It is enough if, for a given basis, every open cover by basis elements has a finite subcover.

If \( A \) is a subspace of \( X \) then compactness of \( A \) is equivalent to the statement that whenever \( A \) is contained in the union of a collection of open subsets of \( X \) then it is contained in the union of a finite subcollection; this follows immediately from the definition of the subspace topology.

If \( f : X \to Y \) is continuous and \( A \subset X \) is compact, then \( f(A) \subset Y \) is compact.

A closed subset of a compact space is always compact.

A compact subset of a Hausdorff space is always closed. Indeed, if \( K \) is compact in \( X \) and \( X \) is Hausdorff then for any point \( p \in X - K \) there are disjoint open sets in \( X \) containing \( p \) and \( K \). (Proof: For \( k \in K \) there are disjoint open sets \( U_k \) and \( V_k \) in \( X \) such that \( p \in U_k \) and \( k \in V_k \). There is a finite set of \( k_i \) such that \( K \) is contained in the union of the \( V_{k_i} \). The point \( p \) belongs to the intersection of the \( U_{k_i} \), which is open, and which is disjoint from the union of the \( V_{k_i} \) and therefore from \( K \).)

A closed interval is compact. (Proof: This is another argument using the completeness of the real number system. Suppose that \([a, b]\) is covered by a collection of open intervals. Let \( T \) be the set of all \( x \in [a, b] \) such that the interval \([a, x]\) is covered by finitely many of those intervals. Let \( c \) be the least upper bound of \( T \). Certainly \( a < c \leq b \). Choose an interval \( J \) in the collection, such that \( c \in J \). Choose \( d \in (a, c) \cap J \cap T \). Then \([a, d]\) is covered by finitely many of the given intervals. If \( c < b \) then those same intervals plus \( J \) cover some interval \([a, c]\) with \( d < e \leq b \), contradicting the choice of \( c \). Thus \( c = b \), and furthermore \( b \in T \) since again those same intervals plus \( J \) cover \([a, b]\).)

The product of two compact spaces \( X \) and \( Y \) is compact. (Proof: Suppose that \( X \times Y \) is covered by \( \{ U_a \times V_a \} \). For each \( x \in X \), \( Y \) is covered by those \( V_a \) for which \( x \in U_a \). There exists a finite set of \( \alpha_i \) (depending on \( x \)), such that \( x \in U_{\alpha_i} \) for all \( i \) and such that \( Y \) is covered by the \( V_{\alpha_i} \). The intersection of the \( U_{\alpha_i} \) is an open set containing \( x \). A finite set of these open sets covers \( X \), and this leads to a finite subcover of the original cover of \( X \times Y \).)

The product of infinitely many compact spaces is also compact, but the proof is harder.

\( \mathbb{R}^n \) topologized in the usual metric fashion is the same as the product of \( n \) copies of the space \( \mathbb{R} \).

It follows that every closed and bounded subset of \( \mathbb{R}^n \) is compact, because it is a closed subset of a rectangle \([-a, a]^n\) and the rectangle is a product of compact intervals.

Conversely, a compact set is closed and bounded. This converse holds in any metric space, not just in \( \mathbb{R}^n \).
A continuous map \( f : X \to Y \) from a compact space \( X \) to a Hausdorff space \( Y \) is always a quotient map. (Proof: \( A \subset X \) closed implies \( A \) compact implies \( f(A) \) compact implies \( f(A) \subset Y \) closed.) In particular \( f \) is a quotient map if it is a surjection, and is a homeomorphism if it is a bijection.