# Wavelets and the angle between past and future

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## 1. Introduction.

In this paper we consider the weighted norm inequalities with matrix weight. Namely, let $W$ be a $d \times d$ matrix weight, i.e. a $L^1$ function whose values are selfadjoint nonnegative $d \times d$ matrices. We suppose that the weight $W$ is defined on a set $\mathcal{R}$ where $\mathcal{R}$ is either the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ or the real line $\mathbb{R}$.

Let $L^2 = L^2(\mathbb{C}^d)$ be the space of square summable functions on $\mathcal{R}$ with values in $\mathbb{C}^d$, $H^2 = H^2(\mathbb{C}^d)$ be the corresponding Hardy space of analytic functions and $P_+$ be the orthogonal projection in $L^2$ onto $H^2$. Let $T$ denote the Hilbert Transform, $T = -iP_+ + i(I - P_+)$. The Hilbert Transform admits well known singular integral representation, for example on the real line $\mathbb{R}$

$$Tf(s) = \frac{1}{\pi}\text{p.v.}\int_{\mathbb{R}} \frac{f(t)}{t-s}dt\,.$$

The question we are interested in is under what conditions on $W$ the following weighted norm inequality for $T$ holds (say for all $f \in L^2 \cap L^\infty$):

$$\int_{\mathcal{R}} (W(t)Tf(t), Tf(t))dt \le C \int_{\mathcal{R}} (W(t)f(t), f(t))dt$$

Clearly this inequality is equivalent to the same inequality for $P_+$ (with another constant) .

If we define a weighted space $L^2(W)$ as the space of all measurable $\mathbb{C}^d$-valued functions on $\mathcal{R}$ satisfying

$$\|f\|^2_{L^2(W)} \stackrel{\text{def}}{=} \int_{\mathcal{R}} (W(t)f(t), f(t))dt < \infty\,,$$

(of course we should factorize it over the subspace of functions of norm 0), then the last inequality means that $T$ (or, equivalently $P_+$) is a bounded operator in $L^2(W)$.

If the dimension $d = 1$ and everything is scalar-valued the answer is given by the famous Hunt — Muckenhoupt — Wheeden theorem, that says that the Muckenhoupt condition $(A_2)$

$$\sup_I \frac{1}{|I|}\int_I W \cdot \frac{1}{|I|} \int_I W^{-1} < \infty$$

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(supremum is taken over all intervals) is necessary and sufficient in this case.

The main result of the article is that the vector Muckenhoupt condition $(\boldsymbol{A}_2)$,

$$\sup_I \left\| \left( \frac{1}{|I|} \int_I W dm \right)^{1/2} \left( \frac{1}{|I|} \int_I W^{-1} dm \right)^{1/2} \right\| < \infty$$

is necessary and sufficient for the boundedness of Hilbert Transform in $L^2(W)$ with matrix weight. This was conjectured by the first author in [4], and the conjecture turns out to be true.

As a byproduct of our techniques we obtain some results on unconditional basis of wavelets in the weighted space $L^2(W)$. We believe that results in such generality are new even in scalar-valued case, as for matrix-valued case we have not seen anything about it in the literature.

In this introduction we shall try to answer the following 3 questions the reader could have already asked him(her)self.

1. Why is this problem interesting ?
2. Why is this problem difficult? Why cannot it be easily reduced to the scalar-valued case?
3. What are connections with other areas?

## 1.1. Origin of the problem.

We think nobody needs to be convinced of the importance of the scalar case (Hunt — Muckenhoupt – Wheeden theorem). One can probably think of many ways to generalize their result to vector case, and we want to convince the reader that our generalization is a "right" one. And we think the simplest way to do that is to explain the origin of the problem.

The problem has two origins (at least for us). The first one is the theory of stationary processes.

Let us consider a multivariate random stationary process. For the simplicity we consider the case of discrete time. Let $\mathcal{W}$ be the spectral measure of the process; in our case it is a measure whose values are $d \times d$ nonnegative selfadjoint matrices. The reader can think of it as of a matrix whose entries are complex measures $\mu_{i,j}$ such that for any Borel set $E$ the matrix $\{\mu_{i,j}(E)\}_{i,j=1}^d$ is nonnegative.

The geometry of the process is described by the geometry of the sequence of subspaces $z^n\mathbb{C}^d$, $n \in \mathbb{Z}$ in a weighted space $L^2(\mathcal{W}) = L^2(\mathcal{W}, \mathbb{C}^d)$. The space $L^2(\mathcal{W})$ consists of all functions on $\mathbb{T}$ with values in $\mathbb{C}^d$ such that

$$\|f\|^2_{L^2(\mathcal{W})} \stackrel{\text{def}}{=} \int_R (d\mathcal{W}(t) f(t), f(t)) < \infty\,.$$

In this representation the past of the process is span $\{z^n\mathbb{C}^d \ : \ n < 0\}$ and the future is span $\{z^n\mathbb{C}^d \ : \ n \geq 0\}$; the angle between past and future is nonzero means simply that the

Riesz projection $P_+$ is bounded in the weighted space $L^2(\mathcal{W})$. This property for stationary Gaussian processes (the angle between past and future is nonzero) is called in probability literature *uniform mixing of past and future.*

If the angle between past and future is nonzero, then for any vector $\boldsymbol{e} \in \mathbb{C}^d$ the angle between subspaces span $\{z^n\boldsymbol{e} \,:\, n < 0\}$ and span $\{z^n\boldsymbol{e} \,:\, n \geq 0\}$ in $L^2(\mathcal{W})$ is nonzero. If a measure $\mu$ is defined on Borel sets $E$ by $\mu(E) = (\mathcal{W}(E)\boldsymbol{e}, \boldsymbol{e})$ the last condition means that the angle between antianalytic polynomials span $\{z^n \,:\, n < 0\}$ and analytic polynomials span $\{z^n \,:\, n \geq 0\}$ in the (scalar) weighted space $L^2(\mu)$ is positive. Equivalently, we can say that the Riesz Projection $P_+$ (or Hilbert Transform $T$) is bounded in the weighted space $L^2(\mu)$. It is well known that this is possible if and only if the measure $\mu$ is absolutely continuous and its density $w$ satisfies the (scalar) Muckenhoupt condition $(A_2)$.

Therefore if the angle between past and future is positive, the spectral measure $\mathcal{W}$ of the process is necessarily absolutely continuous, and the question about the angle gives rise to our problem.

Similarly, if we consider stationary processes with continuous time, we arrive at the problem about Hilbert Transform on $\mathbb{R}$.

Another motivation for the problem comes from the theory of Toeplitz operators. Let us consider a Toeplitz operator $T_F$ with symbol $F$, $T_F : H^2(\mathbb{C}^d) \to H^2(\mathbb{C}^d)$,

$$T_F f = P_+ F f\,, \qquad f \in H^2(\mathbb{C}^d)\,.$$

Here $F$ is a bounded function whose values are $d \times d$ matrices.

It is well known, see [6] that the Toeplitz operator $T_F$ is invertible if and only if its symbol $F$ can be factorized as $F = G_1^* G_2$, where $G_1$, $G_2$ are $H^2$ $d \times d$ matrix-functions ($H^2$ means with entries in $H^2$), such that $G_1^{-1}, G_2^{-1} \in H^2$ and such that the operator

$$G_2^{-1} P_+ (G_1^{-1})^* \tag{1.1}$$

is bounded on $H^2$. We would like to point out that the operator (1.1) is the (formal) inverse to $T_F$, so the above result is very natural.

Since for any $f \in (H^2)^\perp$ we have $G_2^{-1} P_+ (G_1^{-1})^* f = 0$, we can conclude that the operator (1.1) is bounded on $H^2$ if and only if it is bounded on $L^2$. If we denote $W = G_1 G_1^*$, $V = (G_2^{-1})^* G_2^{-1}$ then the boundedness of the operator (1.1) on $L^2$ is equivalent to the following weighted norm inequality

$$\int_{\mathcal{R}} (V(t) P_+ f(t), P_+ f(t)) dt \leq C \int_{\mathcal{R}} (W(t) f(t), f(t)) dt \tag{1.2}$$

It is trivial and well known that given an invertible Toeplitz operator $T_F$, its symbol $F$ has to be invertible in $L^\infty$. Therefore the weights $V$ and $W$ are equivalent in a sense that there exists a constant $C$ such that for all $\boldsymbol{e} \in \mathbb{C}^d$ and for almost all $t \in \mathcal{R}$

$$\frac{1}{C}(V(t)\boldsymbol{e}, \boldsymbol{e}) \leq (W(t)\boldsymbol{e}, \boldsymbol{e}) \leq C(V(t)\boldsymbol{e}, \boldsymbol{e})\,.$$

That means that we can replace in (1.2) $V$ by $W$ or vice versa, so we again arrive at our problem.

## 1.2. Specifics of the matrix-valued case.

We would like to mention that the problem can not be reduced to the scalar case. The proof for the scalar case can not be reproduced to matrix valued weights. The main reason is that the original proof by Hunt — Muckenhoupt — Wheeden [1], and all its modifications (see [2, 3]) heavily use maximal functions. And it is very difficult for us to imagine how can one introduce *working* maximal functions for matrix-valued weights. Indeed, for scalar weighted spaces we have a very simple but wonderful fact that a function $f$ belongs $L^p(\mu)$ if and only if $|f| \in L^p(\mu)$. And we do not have an analogue of this for matrix-valued weights, even for $p = 2$.

As an illustration what kind of difficulties one can encounter in the matrix-valued case let us present a very simple example. It is trivial in the scalar case that if we have an integral operator in $L^2(\mu)$ with positive (scalar) kernel, and we know that an operator with a bigger kernel is bounded, then the original operator is bounded too. This statement (even for scalar kernels) does not hold for weighted $L^2$-spaces with matrix weights (at least we do not know a proof of this fact).

## 1.3. Main idea.

The main idea of the proof is very simple. We choose an appropriate basis and show that the matrix of the operator generates a bounded operator in $\ell^2$. This idea itself is nothing new — it is a standard idea in the wavelet technique.

We would like to point out to two details that distinguish our approach. The first is that the basis system we use is just usual (normalized) Haar system. We believe that for our problem the Haar system is the most appropriate one. Let us mention that at first glance the weighted Haar system like the one used in the solution of Calderon problem [9], or smooth wavelets would be more appropriate. However it is not the case, and the reader can easily see this after he works some time with systems and matrices. In particular, it was shown in [9] that the weighted Haar system forms a Riesz basis in the non-weighted $L^2$, but this is not true for our weights (even in scalar case). In our case the weighted Haar system is not even minimal.

The other distinction is that we do not use the Schur test to prove the boundedness of the operator in the sequence space $\ell^2$. We certainly would prefer to use this test (because it is well known, simple and actually gives necessary and sufficient condition to positive kernels) but we encounter very serious difficulties here. The Schur test multipliers that cancel out in scalar case do not cancel in the matrix case, because of a certain matrix between them. So we have to use another and less known test, the so called Senichkin — Vinogradov test [1] and some of its modifications, see [7] (see also Theorems 6.3, 7.3, 7.4 in the text).

[1] After the paper was submitted it was pointed to us by E. Stein that such "dohbling kernel" trick was used long before by different authors. The paper by A. N. Kolmogorov and G. Silvestrov [11] is the earliest known to us reference

The basic idea of the proof lies entirely in the vein of wavelets theory, but the essential technicalities seem to be new in this context, and we never met them in the literature.

We present the proof only for the real line $\mathbb{R}$, but one can easily see that (almost) the same proof works just fine for the circle $\mathbb{T}$.

Note that we consider only matrix-valued case. We do not know whether the results are true for operator-valued weights ($d = \infty$).

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## 2. Matrix Muckenhoupt condition.

Let $W$ be a matrix-valued weight. We say that the weight $W$ satisfies *matrix Muckenhoupt condition* ($\boldsymbol{A}_2$) (or simply Muckenhoupt condition ($\boldsymbol{A}_2$)) if

$$\sup_I \left\| \left( \frac{1}{|I|} \int_I W dm \right)^{1/2} \left( \frac{1}{|I|} \int_I W^{-1} dm \right)^{1/2} \right\| < \infty$$

where the *supremum* is taken over all intervals $I \subset \mathbb{R}$. If $W$ is a usual scalar function, this condition is simply the well known Muckenhoupt condition $(A_2)$.

In this section we show that this condition is necessary for the Hilbert transform $T$ (or, equivalently, the Riesz Projection $P_+$ to be bounded in the weighted space $L^2(W)$.

Let $W$ be an $d \times d$ matrix weight. Throughout the paper we always assume that the weight $W$ is non-degenerate in the sense that there is no vector $\boldsymbol{e} \in \mathbb{C}^d$ such that $W(t)\boldsymbol{e} = \mathbb{O}$ a.e., because otherwise we can always restrict ourselves to the orthogonal complement of such $\boldsymbol{e}$.

The following lemma holds for an arbitrary measure space $\mathcal{X}$, although we shall apply it for the real line $\mathbb{R}$ or its subsets endowed with Lebesgue measure. For this reason we still use the notation $dt$ for the measure.

Let $k$ be a scalar-valued function in $L^2 \cup L^\infty$, $\|k\|_2 = 1$, $\mathcal{K}$ be a subspace of $L^2(\mathbb{C}^d)$.

$$\mathcal{K} \stackrel{\text{def}}{=} k\mathbb{C}^d = \{k\boldsymbol{e} : \ \boldsymbol{e} \in \mathbb{C}^d\},$$

and let $P$ denotes the orthogonal projection (in the non-weighted space $L^2 = L^2(\mathbb{C}^d)$) onto $\mathcal{K}$. It is easy to see that $P$ is given by the formula

$$Pf = k \cdot \left( \int_{\mathcal{X}} f(t)\overline{k(t)} dt \right) .$$

**Lemma 2.1.** *Given a $d \times d$ matrix weight $W$, the projection $P$ introduced above extends (from a dense set $L^2 \cap L^2(W)$) to a bounded operator on $L^2(W)$ if and only if both weights $W|k|^2$ and $W^{-1}|k|^2$ belong $L^1$. In this case the norm of $P$ in $L^2(W)$ is exactly*

$$\left\| \left[ \int W|k|^2 \right]^{1/2} \left[ \int W^{-1}|k|^2 \right]^{1/2} \right\| .$$

**Proof.** First note that if $P$ extends continuously to a bounded operator in $L^2(W)$ then $W|k|^2 \in L^1$. Indeed, if $W|k|^2 \notin L^1$, one can find a vector $\boldsymbol{e} \in \mathbb{C}^d$ such that

$$\int_{\mathcal{X}} (W(t)\boldsymbol{e}, \boldsymbol{e})|k(t)|^2 dt = \infty .$$

The latter means that $k\boldsymbol{e} \notin L^2(W)$. It is a trivial exercise to find a scalar function $f \in L^2$ such that $f\boldsymbol{e} \in L^2(W)$ and $(f, k)_{L^2} = \alpha \neq 0$. Then $Pf\boldsymbol{e} = \alpha k\boldsymbol{e} \notin L^2(W)$, and so $P$ is not bounded in $L^2(W)$.

Let now $W|k|^2 \in L^1$. Denote by $\mathcal{L}$ the orthogonal complement of $\mathcal{K}$ in $L^2$, and let $\mathcal{L}_0 \stackrel{\text{def}}{=} \mathcal{L} \cap L^2(W)$. For any vector $\boldsymbol{e} \in \mathbb{C}^d$

$$\begin{aligned} \operatorname{dist}_{L^2(W)}\{k\boldsymbol{e}, \mathcal{L}_0\} &= \operatorname{dist}_{L^2}\{W^{1/2}k\boldsymbol{e}, W^{1/2}\mathcal{L}_0\} = \\ &= \sup\left\{ (W^{1/2}k\boldsymbol{e}, g) \ : \ g \in (W^{1/2}\mathcal{L}_0)^{\perp}, \ \|g\| = 1 \right\} . \end{aligned}$$

Suppose for a moment that the weight $W^{-1}$ exists and is uniformly bounded. Then it is easy to see that $(W^{1/2}\mathcal{L}_0)^{\perp} = W^{-1/2}\mathcal{K}$ So we can continue the above chain of equalities

$$\begin{aligned} &\operatorname{dist}_{L^2(W)}(k\boldsymbol{e}, \mathcal{L}_0) = \sup\left\{ \left|\left(W^{1/2}k\boldsymbol{e}, W^{-1/2}k\boldsymbol{f}\right)_{L^2}\right| \ : \ \boldsymbol{f} \in \mathbb{C}^d , \ \|W^{-1/2}k\boldsymbol{f}\| \leq 1 \ \right\} = \\ = \quad &\sup\left\{ |(k\boldsymbol{e}, k\boldsymbol{f})_{L^2}| \ : \ \boldsymbol{f} \in \mathbb{C}^d , \ \int_{\mathbb{R}} (W^{-1}(t)\boldsymbol{f}, \boldsymbol{f})\, |k(t)|^2 \, dt \leq 1 \ \right\} = \\ = \quad &\sup\left\{ |(\boldsymbol{e}, \boldsymbol{f})_{L^2}| \ : \ \boldsymbol{f} \in \mathbb{C}^d , \ (\left[\int W^{-1}|k|^2\right] \boldsymbol{f}, \boldsymbol{f}) \leq 1 \ \right\} = \\ = \quad &\sup\left\{ \left|(\boldsymbol{e}, \left[\int W^{-1}|k|^2\right]^{-1/2} \boldsymbol{g})_{L^2}\right| \ : \ \boldsymbol{g} \in \mathbb{C}^d , \ \|\boldsymbol{g}\| \leq 1 \ \right\} = \left\| \left[\int W^{-1}|k|^2\right]^{-1/2} \boldsymbol{e} \right\| \end{aligned}$$

So we have proved that under the assumption $W^{-1} \in L^{\infty}$

$$\operatorname{dist}_{L^2(W)}(k\boldsymbol{e}, \mathcal{L}_0) = \left\| \left[\int W^{-1}|k|^2\right]^{-1/2} \boldsymbol{e} \right\| . \tag{2.1}$$

Let now $W$ be a general weight. Define $W_\varepsilon \stackrel{\text{def}}{=} W + \varepsilon I$. As it was shown above (2.1) holds for $W_\varepsilon$. Since for any $f \in \mathcal{K} + \mathcal{L}_0$ its weighted norm $\|f\|_{L^2(W_\varepsilon)}$ decreases as $\varepsilon$ decreases, and has limit $\|f\|_{L^2(W)}$ as $\varepsilon \to 0$, it follows that

$$\operatorname{dist}_{L^2(W)}(k\boldsymbol{e}, \mathcal{L}_0) = \lim_{\varepsilon \to 0+} \operatorname{dist}_{L^2(W_\varepsilon)}(k\boldsymbol{e}, \mathcal{L}_0) = \lim_{\varepsilon \to 0+} \left[\int_{\mathbb{R}} (W_\varepsilon^{-1}(t)\boldsymbol{e}, \boldsymbol{e})|k|^2 \, dt\right]^{1/2}$$

This implies that if $\operatorname{dist}_{L^2(W)}(k\boldsymbol{e}, \mathcal{L}_0)$ is nonzero for all nonzero $\boldsymbol{e}$, then $W^{-1}|k|^2 \in L^1$ and formula (2.1) holds. But, as it will be shown below, that is exactly the case when $P$ admits continuous extension to $L^2(W)$.

It follows from elementary geometry of Hilbert spaces that

$$\|P\|_{L^2(W)} = \sup\left\{ \frac{\|k\boldsymbol{e}\|_{L^2(W)}}{\operatorname{dist}_{L^2(W)}(k\boldsymbol{e}, \mathcal{L}_0)} \ : \ \boldsymbol{e} \in \mathbb{C}^d, \ \boldsymbol{e} \neq 0 \ \right\} =$$

$$= \sup \left\{ \frac{\left\| [\int W|k|^2]^{1/2} \boldsymbol{e} \right\|}{\left\| [\int W^{-1}|k|^2]^{-1/2} \boldsymbol{e} \right\|} \ : \ \boldsymbol{e} \in \mathbb{C}^d, \ \boldsymbol{e} \neq 0 \right\} =$$

$$= \sup \left\{ \left\| \left[ \int W|k|^2 \right]^{1/2} \left[ \int W^{-1}|k|^2 \right]^{1/2} \boldsymbol{f} \right\| \ : \ \boldsymbol{f} \in \mathbb{C}^d, \ \|\boldsymbol{f}\| = 1 \right\} =$$

$$= \left\| \left[ \int W|k|^2 \right]^{1/2} \left[ \int W^{-1}|k|^2 \right]^{1/2} \right\| .$$

and therefore the part *"only if"* of the lemma is proved.

To prove the *"if"* part of the lemma one has to reverse the reasoning above. □

Let $k_\lambda$ be the normalized reproducing kernel

$$k_\lambda \stackrel{\text{def}}{=} \left( \frac{2 \operatorname{Im} \lambda}{\pi} \right)^{1/2} \frac{1}{(z - \overline{\lambda})}, \qquad \lambda \in \mathbb{C}_+ .$$

Note that $|k_\lambda(t)|^2$ is the Poisson kernel for $\mathbb{C}_+$, i.e. the harmonic extension $W(\lambda)$ of a function $W$ at the point $\lambda \in \mathbb{C}_+$ is given by the formula

$$W(\lambda) = \int_{\mathbb{R}} W(t) |k_\lambda(t)|^2 dt .$$

**Lemma 2.2.** *Assume the Riesz projection $P_+$ is bounded in the weighted space $L^2(W)$. Then*

$$\sup_{\lambda \in \mathbb{C}_+} \left\| \left[ W(\lambda) \right]^{1/2} \left[ W^{-1}(\lambda) \right]^{1/2} \right\| < \infty \tag{2.2}$$

*where $W(\lambda)$, $W^{-1}(\lambda)$ denote harmonic extensions at the point $\lambda$ of $W$ and $W^{-1}$ respectively. The latter condition in its turn implies the matrix Muckenhoupt condition*

$$\sup_I \left\| \left( \frac{1}{|I|} \int_I W dm \right)^{1/2} \left( \frac{1}{|I|} \int_I W^{-1} dm \right)^{1/2} \right\| < \infty \tag{2.3}$$

*where* supremum *is taken over all intervals $I \subset \mathbb{R}$.*

**Proof.** Define a subspace $\mathcal{K}_\lambda$,

$$\mathcal{K}_\lambda \stackrel{\text{def}}{=} k_\lambda \mathbb{C}^d = \{ k_\lambda \boldsymbol{e} : \ \boldsymbol{e} \in \mathbb{C}^d \} ,$$

and let $P_\lambda$ denotes the orthogonal projection (in the non-weighted space $L^2 = L^2(\mathbb{C}^d)$) onto $\mathcal{K}_\lambda$. The operator $P_\lambda$ can be represented as

$$P_\lambda f = P_+ f - b_\lambda P_+ \overline{b}_\lambda f , \qquad f \in L^2(\mathbb{C}^d) ,$$

where $b_\lambda$ stands for an elementary Blaschke factor with zero at $\lambda \in \mathbb{C}_+$, $b_\lambda \stackrel{\text{def}}{=} (z - \lambda)/(z - \overline{\lambda})$.

Since $P_+$ is bounded in $L^2(W)$ and the multiplication by a unimodular function $b_\lambda$ is a unitary operator there, operators $P_\lambda$, $\lambda \in \mathbb{C}_+$ are uniformly bounded. Applying Lemma 2.1 and taking into account that

$$\int W|k_\lambda|^2 = W(\lambda)\,, \qquad \text{and} \qquad \int W^{-1}|k_\lambda|^2 = W^{-1}(\lambda)\,,$$

we can obtain (2.2).

Condition (2.2) can be rewritten as

$$\sup_{\lambda\in\mathbb{C}_+} \sup \left\{ \frac{\left\| [W(\lambda)]^{1/2}\boldsymbol{e}\right\|}{\left\| [W^{-1}(\lambda)]^{-1/2}\boldsymbol{e}\right\|} \ : \ \boldsymbol{e}\in\mathbb{C}^d,\ \boldsymbol{e}\neq 0 \right\} < \infty\,. \tag{2.4}$$

Fix a point $\lambda \in \mathbb{C}_+$ and denote by $I = I_\lambda$ the interval $[\operatorname{Re}\lambda - \operatorname{Im}\lambda,\, \operatorname{Re}\lambda + \operatorname{Im}\lambda] \subset \mathbb{R}$. Since for the Poisson $\mathcal{P}_\lambda(t)$ kernel we have on the interval $I$

$$\mathcal{P}_\lambda(t) \ = \ \frac{\operatorname{Im}\lambda}{\pi|t-\overline{\lambda}|^2} \geq \frac{1}{\pi\operatorname{Im}\lambda} \ = \ \frac{2}{\pi|I|}\,,$$

the following inequality holds for any matrix weight $W$:

$$W_I \ \stackrel{\text{def}}{=} \ \frac{1}{|I|}\int_I W \leq \frac{\pi}{2}W(\lambda)\,, \qquad I = I_\lambda\,;$$

the inequality means usual inequality for quadratic forms. Therefore for $I = I_\lambda$

$$\frac{(W_I\boldsymbol{e},\boldsymbol{e})}{\left([(W^{-1})_I]^{-1}\boldsymbol{e},\boldsymbol{e}\right)} \ \leq \ \left(\frac{\pi}{2}\right)^2 \frac{(W(\lambda)\boldsymbol{e},\boldsymbol{e})}{\left([W^{-1}(\lambda)]^{-1}\boldsymbol{e},\boldsymbol{e}\right)} \tag{2.5}$$

Since

$$\left\|(W_I)^{1/2}\left((W^{-1})_I\right)^{1/2}\right\| \ = \ \sup\left\{ \frac{\left\|[W_I]^{1/2}\boldsymbol{e}\right\|}{\left\|[(W^{-1})_I]^{-1/2}\boldsymbol{e}\right\|} \ : \ \boldsymbol{e}\in\mathbb{C}^d,\ \boldsymbol{e}\neq 0 \right\} \ < \ \infty\,,$$

inequalities (2.4) and (2.5) imply (2.3). □

## 3. Properties of matrix Muckenhoupt weights.

First we will prove some lemmas about general (not necessary Muckenhoupt) matrix weights.

**Lemma 3.1.** *Let $W$ be a nonnegative, measurable, $d\times d$ matrix function on a measure space $\mathcal{X}$, $\mu$. Then*

$$\int_{\mathcal{X}} \|W(t)\|d\mu(t) \leq d\left\|\int_{\mathcal{X}} W(t)d\mu(t)\right\|$$

**Proof.** Since for a nonnegative $d \times d$ matrix $A$ we have $\|A\| \le \operatorname{trace} A \le d\|A\|$, we can conclude that

$$\int_{\mathcal{X}} \|W(t)\| d\mu(t) \le \int_{\mathcal{X}} \operatorname{trace}(W(t)) d\mu(t) = \operatorname{trace}\left(\int_{\mathcal{X}} W(t) d\mu(t)\right) \le d \cdot \left\| \int_{\mathcal{X}} W(t) d\mu(t) \right\| .$$

□

Given a matrix weight $W$ and a set $I \subset \mathbb{R}$ let us denote by $W(I)$ the integral $W(I) \stackrel{\text{def}}{=} \int_I W(t)dt$ and by $W_I$ the average value $W_I \stackrel{\text{def}}{=} W(I)/|I|$.

**Lemma 3.2.** *Let $W$ be a $d \times d$ matrix-valued weight such that $W$ and $W^{-1}$ are summable on a measurable set $I$. Then for any vector $\boldsymbol{e} \in \mathbb{C}^d$*

$$\frac{(W_I \boldsymbol{e}, \boldsymbol{e})}{\left(\left[(W^{-1})_I\right]^{-1} \boldsymbol{e}, \boldsymbol{e}\right)} \ge 1 .$$

**Proof.** The statement of the lemma has very simple geometrical interpretation. The numerator $(W_I \boldsymbol{e}, \boldsymbol{e})$ is exactly the norm of a function identically equal $\boldsymbol{e}$ in the weighted space $L^2(I, W)$, and the denominator $\left(\left[(W^{-1})_I\right]^{-1} \boldsymbol{e}, \boldsymbol{e}\right)$ is the distance (in the same space) from this function to the space of all (vector)-functions satisfying $\int_I f(t)dt = 0$. The formal proof that we preset below, is, in essence, contained in the proof of Lemma 2.1. The vector $\boldsymbol{f}$ that we use here is the same vector $\boldsymbol{f}$ from the proof of Lemma 2.1 (up to the notation).

Define a vector $\boldsymbol{f} = [W^{-1}(I)]^{-1} \boldsymbol{e}$. Then

$$\begin{aligned} |I| \cdot \left(\left[W^{-1}(I)\right]^{-1} \boldsymbol{e}, \boldsymbol{e}\right) = |I|(e, f) = \int_I \left(W^{1/2}(t)\boldsymbol{e}, W^{-1/2}(t)\boldsymbol{f}\right) dt \le \\ \le \left(\int_I (W(t)\boldsymbol{e}, \boldsymbol{e})\, dt)\right)^{1/2} \left(\int_I \left(W^{-1}(t)\boldsymbol{f}, \boldsymbol{f}\right) dt\right)^{1/2} = \\ = (W(I)\boldsymbol{e}, \boldsymbol{e})^{1/2} \left(W^{-1}(I)\boldsymbol{f}, \boldsymbol{f}\right)^{1/2} = (W(I)\boldsymbol{e}, \boldsymbol{e})^{1/2} \left(\left[W^{-1}(I)\right]^{-1} \boldsymbol{e}, \boldsymbol{e}\right)^{1/2} \end{aligned}$$

Therefore

$$|I|^2 \le \frac{(W(I)\boldsymbol{e}, \boldsymbol{e})}{\left([W^{-1}(I)]^{-1} \boldsymbol{e}, \boldsymbol{e}\right)} .$$

The latter inequality is equivalent to the statement of the lemma. □

**Corollary 3.3.** *For the matrix weight $W$ from Lemma 3.2 above the operator $A = [W_I]^{1/2}[(W^{-1})_I]^{1/2}$ is an expanding operator, i.e. $\|Ae\| \ge \|e\|$ for all vectors $e$.*

**Proof.** Follows immediately from Lemma 3.2. □

And now some properties of $(\boldsymbol{A}_2)$ weights. First of all note that $(\boldsymbol{A}_2)$ condition is symmetric, that means a matrix weight $W$ satisfies $(\boldsymbol{A}_2)$ if and only if $W^{-1}$ does.

It is well-known that usual scalar $A_2$-weights satisfy the doubling property. A similar property holds for matrix $\boldsymbol{A}_2$-weights.

**Lemma 3.4.** *Let $W$ satisfy the vector condition ($\boldsymbol{A}_2$). There exists a constant $C = C(W)$ such that for any $\boldsymbol{e} \in \mathbb{C}^d$ and any intervals $I \subset J$, $|J| \le 2|I|$, the following inequality*

$$\int_J (W(t)\boldsymbol{e}, \boldsymbol{e})dt \le C \int_I (W(t)\boldsymbol{e}, \boldsymbol{e})dt.$$

*holds.*

**Proof.** Lemma 3.2 implies the first inequality in the following chain:

$$|I|^2 \le \frac{(W(I)\boldsymbol{e}, \boldsymbol{e})}{([W^{-1}(I)]^{-1}\boldsymbol{e}, \boldsymbol{e})} \le \frac{(W(J)\boldsymbol{e}, \boldsymbol{e})}{([W^{-1}(I)]^{-1}\boldsymbol{e}, \boldsymbol{e})} \le \frac{(W(J)\boldsymbol{e}, \boldsymbol{e})}{([W^{-1}(J)]^{-1}\boldsymbol{e}, \boldsymbol{e})} \le A|J|^2 \le 4A|I|^2$$

(the fourth inequality in the chain is simply a Muckenhoupt condition). So

$$|I|^2 \le \frac{(W(I)\boldsymbol{e}, \boldsymbol{e})}{([W^{-1}(I)]^{-1}\boldsymbol{e}, \boldsymbol{e})} \le \frac{(W(J)\boldsymbol{e}, \boldsymbol{e})}{([W^{-1}(I)]^{-1}\boldsymbol{e}, \boldsymbol{e})} \le 4A|I|^2$$

and hence

$$\frac{(W(J)\boldsymbol{e}, \boldsymbol{e})}{(W(I)\boldsymbol{e}, \boldsymbol{e})} \le 4A\,.$$

□

**Lemma 3.5.** *Let $W$ be a $d \times d$ matrix Muckenhoupt weight, and $A$ be a positive nonsingular $d \times d$ (constant) matrix. Then the weight $\mathcal{W} \stackrel{\text{def}}{=} AWA$ also satisfies the Muckenhoupt condition ($\boldsymbol{A}_2$) with the same constant.*

**Proof.** Clearly, $\mathcal{W}_I = AW_IA$ and $(\mathcal{W}^{-1})_I = A^{-1}(W^{-1})_I A^{-1}$. Operator $B = (W_I)^{1/2} A$ is a Hermitian square root of $\mathcal{W}_I$, that means $B^*B = \mathcal{W}_I$. Therefore we can write a polar decomposition for the operator $B = (W_I)^{1/2} A = U (\mathcal{W}_I)^{1/2}$, where $U$ is a unitary matrix. Similarly, for an operator $C = ((W^{-1})_I)^{1/2} A^{-1}$ one can write $C = V ((\mathcal{W}^{-1})_I)^{1/2}$, where $V$ again is a unitary matrix.

Therefore

$$BC^* = (W_I)^{1/2} \left((W^{-1})_I\right)^{1/2} = U (\mathcal{W}_I)^{1/2} \left((\mathcal{W}^{-1})_I\right)^{1/2} V^*\,,$$

and so

$$\left\|(W_I)^{1/2} \left((W^{-1})_I\right)^{1/2}\right\| = \left\|(\mathcal{W}_I)^{1/2} \left((\mathcal{W}^{-1})_I\right)^{1/2}\right\|\,.$$

□

**Lemma 3.6.** *For a $d \times d$ matrix Muckenhoupt weight $W$ the following inequality holds;*

$$\frac{1}{|I|}\int_I \|W\| \le C\left(\frac{1}{|I|}\int_I \|W\|^{1/2}\right)^2$$

*(Clearly, since $d < \infty$ the norm here can be replaced by trace, Hilbert–Schmidt norm, or any $S_p$ norm.)*

**Proof.** One can easily see from Lemma 2.1 (with $k = |I|^{-1/2}\chi_I$) that the Muckenhoupt condition is equivalent to the uniform boundedness of all averaging operators

$$f \mapsto \chi_I \frac{1}{|I|}\int_I f, \qquad f \in L^2(\mathbb{C}^d)\,.$$

Fix any direction (unit vector) $\boldsymbol{e} \in \mathbb{C}^d$, and consider the restrictions of these operators on the subspace $\{f\boldsymbol{e} \ : \ f \in L^2\}$. It follows that for any $\boldsymbol{e} \in \mathbb{C}^d$, $\|\boldsymbol{e}\| = 1$ the scalar weight $w(t) = (W(t)\boldsymbol{e}, \boldsymbol{e})$ is a Muckenhoupt weight with the same or better constant as $W$. So any diagonal element of $W$ is a Muckenhoupt weight (with the control over the constant).

It is very easy to see, that if $d$ scalar weights satisfy the Muckenhoupt condition with the constant at most $C$, then their sum also satisfy the Muckenhoupt condition with the constant at most $dC$. Therefore $\operatorname{trace} W$ is a Muckenhoupt weight.

The conclusion of the lemma is well known (and now almost trivial) for scalar weights, so

$$\frac{1}{|I|}\int_I \operatorname{trace} W \le C\left(\frac{1}{|I|}\int_I (\operatorname{trace} W)^{1/2}\right)^2$$

Since $d < \infty$ we can replace trace by the norm. $\square$

# 4. A Carleson measure.

The main result of this section is the following theorem about matrix Muckenhoupt weights. Given a dyadic interval $I$ let $Q_I$ denote the corresponding "Carleson square", $Q_I \stackrel{\text{def}}{=} \{z \in \mathbb{C}_+ \ : \ \operatorname{Re} z \in I,\ 0 \le \operatorname{Im} z \le |I|\}$, and let $\lambda(I)$ be the middle point of the upper side of $Q_I$, i.e. $\operatorname{Im}\lambda(I) = |I|$ and $\operatorname{Re}\lambda(I)$ be the center of $I$. Let $\delta_\lambda$ denotes the unit mass at the point $\lambda$. By $I_1$ and $I_2$ we denote the left and the right halves of $I$ respectively.

**Theorem 4.1.** *Let $W$ be a $d\times d$ Muckenhoupt weight. Then the measure*

$$M = \sum_{I\in\mathcal{D}} \left\|(W_I)^{-1/2}(W_{I_1} - W_{I_2})(W_I)^{-1/2}\right\|^2 |I|\delta_{\lambda(I)}$$

*is Carleson. That means there exists a constant $C$ such that*

$$\sum_{I\in\mathcal{D},\, I\subset J} \left\|(W_I)^{-1/2}(W_{I_1} - W_{I_2})(W_I)^{-1/2}\right\|^2 |I| \le C|J| \tag{4.1}$$

*for all dyadic intervals $J$*

**Remark.** Formally speaking, the theorem claims that the measure $M$ is a dyadic Carleson measure, since in the definition of Carleson measure the interval $J$ in (4.1) is an arbitrary, not necessarily dyadic, interval. But any interval can be covered by at most 2 dyadic intervals of lengths comparable with the length of the interval itself, so a dyadic Carleson measure is a usual Carleson measure.

For $(A_\infty)$ scalar weights this was proved in [10]. Our proof is formally different. We do not touch $(A_\infty)$ matrix weights, but for scalar $(A_\infty)$ weights our proof also works.

To prove the theorem we need some lemmas.

**Lemma 4.2.** *Let $A$ and $B$ be nonsingular positive $d \times d$ matrices. Then*

$$\sqrt{\det A \det B} \le \det\left(\frac{A+B}{2}\right)$$

**Proof.** If the lemma holds for some matrices $A$ and $B$, it holds for all matrices of form $C^*AC$, $C^*BC$ — this is because determinant is multiplicative. Taking $C = [(A+B)/2]^{-1/2}$ one can see that it is sufficient to prove the lemma only in the case $(A+B)/2 = \boldsymbol{I}$.

Let now $(A+B)/2 = \boldsymbol{I}$. Then $A = \boldsymbol{I} + \Delta$, $B = \boldsymbol{I} - \Delta$, $\Delta = \Delta^*$. Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be eigenvalues of $\Delta$. Then the eigenvalues of $A$ and $B$ are $1+\alpha_1, 1+\alpha_2, \ldots, 1+\alpha_d$ and $1-\alpha_1, 1-\alpha_2, \ldots, 1-\alpha_d$ respectively. Since $(1+\alpha_k)(1-\alpha_k) = 1-\alpha_k^2 \le 1$,

$$\det(AB) = \det A \det B = \prod_{k=1}^d (1+\alpha_k)(1-\alpha_k) \le 1 = \left[\det\left(\frac{A+B}{2}\right)\right]^{1/2}$$

□

**Proof of Theorem 4.1.** For a dyadic interval $I$ let us denote

$$\mu(I) \stackrel{\text{def}}{=} \det W_I\,, \quad \nu(I) \stackrel{\text{def}}{=} \det(W^{-1})_I\,, \quad m(I) \stackrel{\text{def}}{=} \mu(I)\nu(I) = \det\left((W_I)^{1/2}(W_I^{-1})(W_I)^{1/2}\right).$$

Let $I_1$ and $I_2$ be the right and left halves of the interval $I$. Since $W_I = (W_{I_1} + W_{I_2})/2$, Lemma 4.2 implies $\mu(I)^2 \ge \mu(I_1)\mu(I_2)$ and similarly, $\nu(I)^2 \ge \nu(I_1)\nu(I_2)$. Let us estimate the ratio

$$\frac{\mu(I_1)\mu(I_2)}{\mu(I)^2} = \frac{\det W_{I_1} \det W_{I_2}}{(\det W_I)^2} = \det\left((W_I)^{-1/2} W_{I_1} (W_I)^{-1/2}\right) \det\left((W_I)^{-1/2} W_{I_2} (W_I)^{-1/2}\right)$$

Denote

$$A \stackrel{\text{def}}{=} (W_I)^{-1/2} W_{I_1} (W_I)^{-1/2}\,, \qquad B \stackrel{\text{def}}{=} (W_I)^{-1/2} W_{I_2} (W_I)^{-1/2}\,.$$

Since $(A+B)/2 = I$, we can write $A = I + \Delta$, $B = I - \Delta$. Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be the eigenvalues of $\Delta$, then

$$\det A \det B = \prod_{k=1}^d (1+\alpha_k) \cdot \prod_{k=1}^d (1-\alpha_k) = \prod_{k=1}^d (1-\alpha_k^2) = \exp\left(\sum_{k=1}^d \log(1-\alpha_k^2)\right) \le$$
$$\le \exp\left(-\sum_{k=1}^d \alpha_k^2\right) = \exp\left(-\operatorname{trace}(\Delta^2)\right) = \exp\left(-\frac{1}{4}\operatorname{trace}((A-B)^2)\right)$$

So we have proved that

$$\mu(I) \ge (\mu(I_1)\mu(I_2))^{1/2} \exp\left\{\frac{1}{2}\cdot\frac{1}{4}\operatorname{trace}\left(\left((W_I)^{-1/2}(W_{I_1} - W_{I_2})(W_I)^{-1/2}\right)^2\right)\right\}$$

Taking into account that $\nu(I)^2 \geq \nu(I_1)\nu(I_2)$ we get

$$m(I) \geq (m(I_1)m(I_2))^{1/2} \exp\left\{\frac{1}{2} \cdot \frac{1}{4}\text{trace}\left(\left[(W_I)^{-1/2}(W_{I_1} - W_{I_2})(W_I)^{-1/2}\right]^2\right)\right\} \tag{4.2}$$

(let us remind that $m(I) \stackrel{\text{def}}{=} \mu(I)\nu(I)$ ).

Applying (4.2) to $I_1$, $I_2$, then to halves of these intervals we get on the $n$th step

$$m(I) \geq \left[\prod_{\substack{J\in\mathcal{D}\,:\,J\subset I,\\ |J|=|I|2^{-n}}} m(J)\right]^{1/2^n} \exp\left(\frac{1}{8}\sum_{\substack{J\in\mathcal{D}\,:\,J\subset I,\\ |J|>|I|2^{-n}}} \frac{|J|}{|I|}\text{trace}\left(\left[(W_J)^{-1/2}(W_{J_1} - W_{J_2})(W_J)^{-1/2}\right]^2\right)\right)$$

Since by Corollary 3.3 operators $(W_J)^{1/2}((W^{-1})_J)^{1/2}$ are expanding, it follows that $m(J) \geq 1$. On the other hand $m(I) \leq C$ (Muckenhoupt condition), so

$$\sum_{\substack{J\in\mathcal{D}\,:\,J\subset I,\\ |J|>|I|2^{-n}}} \frac{|J|}{|I|}\text{trace}\left(\left[(W_J)^{-1/2}(W_{J_1} - W_{J_2})(W_J)^{-1/2}\right]^2\right) \leq C$$

for all $n$ ($C$ does not depend on $n$ and $I$). The latter is exactly the conclusion of the lemma. □

# 5. Construction of a vector Haar system.

## 5.1. Preliminaries about bases of subspaces.

Consider a system of subspaces $E_n$ in a Hilbert $\mathcal{H}$ space. For simplicity let us suppose that the system is complete, i.e. the only vector orthogonal to all $E_n$ is 0.

Let us recall that the system is called *minimal* if there exists a family of bounded (not necessary orthogonal) projections $\mathcal{E}^n$

$$\mathcal{E}^n f = \begin{cases} f, & f \in E_n \\ 0, & f \in E_k\,, k \neq n\,. \end{cases}$$

and is called uniformly minimal if projections $\mathcal{E}^n$ are uniformly bounded.

For a minimal system of subspaces $E_n$ one can define the so-called *biorthogonal* or *dual* system $E'_n \stackrel{\text{def}}{=} (\mathcal{E}^n)^*\mathcal{H}$. Clearly $E'_n = \text{span}\,\{E_k \ : \ k \neq n\}^\perp$.

A complete system of subspaces $E_n$ is called a *Riesz basis* or *unconditional basis* if there exists an isomorphism from $\mathcal{H}$ onto another Hilbert space $\mathcal{H}'$ that maps the system $E_n$ into an orthogonal system. Such an isomorphism is called the *orthogonalizer* of the system. Another equivalent definition: a system of subspaces $E_n$ is a Riesz basis if there exists a constant $C$ such that for any (finite) collection of vectors $f_n \in E_n$

$$\frac{1}{C}\sum \|f_k\|^2 \leq \left\|\sum f_k\right\|^2 \leq C\sum \|f_k\|^2$$

A system of vectors $f_n$ is said to be an *unconditional basis* if the corresponding system of one-dimensional spaces is an unconditional basis, and we call the system of vectors a Riesz basis if, in addition, it is almost normalized, i.e.

$$0 < \inf_n \|f_n\| \le \sup_n \|f_n\| < \infty\,.$$

In this case the norm $\|\sum c_k f_k\|$ is equivalent to $\ell^2$-norm of the sequence $\{c_n\}_n$.

We need the following well known fact about Riesz bases, see [7].

**Theorem 5.1** *A complete system of subspaces $E_n$ is a Riesz basis if and only if it is uniformly minimal and the following two "imbedding theorems" hold:*

1. $\sum \|P_{E_n} f\|^2 \le C\|f\|^2$ *for all* $f \in \mathcal{H}$*;*
2. $\sum \|P_{E_n'} f\|^2 \le C\|f\|^2$ *for all* $f \in \mathcal{H}$*.*

## 5.2. Construction of a vector Haar system.

Let $\mathcal{D}$ denote the set of all dyadic intervals on $\mathbb{R}$. Given a dyadic interval $I \in \mathcal{D}$ let $I_1$ and $I_2$ denote its left and right halves respectively, and let $h_I$ be a scalar Haar function, $h_I = |I|^{-1/2}(\chi_{I_2} - \chi_{I_1})$. Denote by $\boldsymbol{e}_I^k$, $k = 1, 2, ..., d$ and orthonormal basis of eigenvalues of $W(I)$ (or, equivalently of $W_I$) in $\mathbb{C}^d$. Let

$$w_I^k = \left(\frac{1}{|I|}\int_I (W(t)\boldsymbol{e}_I^k, \boldsymbol{e}_I^k)dt\right)^{-1/2} = \left[(W_I\boldsymbol{e}_I^k, \boldsymbol{e}_I^k)\right]^{-1/2} = \left[([W_I]^{-1}\boldsymbol{e}_I^k, \boldsymbol{e}_I^k)\right]^{1/2} = \left\|[W_I]^{-1/2}\boldsymbol{e}_I^k\right\|\,.$$

Then the system

$$\boldsymbol{f}_I^k = w_I^k h_I \boldsymbol{e}_I^k\,, \qquad I \in \mathcal{D},\ k = 1, 2, ..., d$$

is a normalized ($\|\boldsymbol{f}_I^k\|_{L^2(W)} = 1$) complete system in $L^2(W)$.

Let us recall that a complete system of vectors $f_n$ in a Hilbert space is called a Riesz basis if there exists a constant $C > 0$ such that

$$\frac{1}{C}\sum_n |c_n|^2 \le \|\sum_n c_n f_n\|^2 \le C\sum_n |c_n|^2$$

for all finitely supported sequences $c_n$.

**Theorem 5.2.** *Let the weight $W$ satisfy vector Muckenhoupt condition ($\boldsymbol{A}_2$). Then the system*

$$\boldsymbol{f}_I^k = w_I^k h_I \boldsymbol{e}_I^k\,, \qquad I \in \mathcal{D},\ k = 1, 2, ..., d$$

*is a Riesz basis in $L^2(W)$.*

Note that if the weight $W$ satisfies the Muckenhoupt condition ($\boldsymbol{A}_2$), the the system $E_I$ is uniformly minimal. Indeed, the subspaces $E_I$ are orthogonal in the non-weighted space

$L^2$, so the corresponding projections $\mathcal{P}^I$ are simply orthogonal (in the non-weighted $L^2$) projections onto $E_I$. By Lemma 2.1 the norm of $\mathcal{P}^I$ is

$$\left\| \left(\frac{1}{|I|}\int_I W\right)^{1/2} \left(\frac{1}{|I|}\int_I W^{-1}\right)^{1/2} \right\| \leq C < \infty$$

hence the system is uniformly minimal.

**Proof of Theorem 5.2.** Vectors $f_I^k$, $k = 1, 2, \ldots, d$ form an orthonormal (in $L^2(W)$) basis in $E_I$, and hence we know that to prove the theorem it is sufficient to show that the system of subspaces $E_I$ forms a Riesz basis. Since we already now that the system of subspaces is uniformly minimal, it is sufficient to show that two imbedding theorems hold.

The system of functions $E_I$ in the weighted space $L^2(W)$ has the same geometry as the system $W^{1/2}E_I$ in the non-weighted space $L^2$. It is easy to see that the biorthogonal to the latter system is $W^{-1/2}E_I$. So, due to the symmetry of the Muckenhoupt condition we only need to check one imbedding theorem.

To show that the imbedding theorem holds we need a modified Haar system which is orthogonal in the weighted space $L^2(W)$. Namely, there exist vectors $\tilde{\boldsymbol{e}}_I^k \in \mathbb{C}^d$ such that the system

$$g_I^k \stackrel{\text{def}}{=} w_I^k h_I \boldsymbol{e}_I^k + \chi_I \tilde{\boldsymbol{e}}_I^k$$

is orthogonal in $L^2(W)$. Clearly, one only has to check that for all $I$ and $k$

$$\int_I (W(t) g_I^k(t), \boldsymbol{f}) dt = 0 \qquad \forall \boldsymbol{f} \in \mathbb{C}^d$$

One can rewrite the above condition as

$$0 = |I|^{-1/2} w_I^k \left[ (W(I_2)\boldsymbol{e}_I^k, \boldsymbol{f}) - (W(I_1)\boldsymbol{e}_I^k, \boldsymbol{f}) \right] - (W(I)\tilde{\boldsymbol{e}}_I^k, \boldsymbol{f}),$$

and since this should hold for all $\boldsymbol{f} \in \mathbb{C}^d$ it follows that

$$\tilde{\boldsymbol{e}}_I^k = |I|^{-1/2} w_I^k W(I)^{-1}(W(I_1) - W(I_2))\boldsymbol{e}_I^k .$$

Taking into account that $w_I^k = \left\| [W_I]^{-1/2}\, \boldsymbol{e}_I^k \right\|$ and $\boldsymbol{e}_I^k$ is an eigenvector of $W_I$ we can rewrite

$$\begin{aligned} \tilde{\boldsymbol{e}}_I^k &= |I|^{-1/2} W(I)^{-1}(W(I_1) - W(I_2))(W_I)^{-1/2}\boldsymbol{e}_I^k = \\ &= \frac{1}{2}|I|^{-1/2} W_I^{-1}(W_{I_1} - W_{I_2})(W_I)^{-1/2}\boldsymbol{e}_I^k \stackrel{\text{def}}{=} A_I \boldsymbol{e}_I^k , \end{aligned}$$

where

$$A_I \stackrel{\text{def}}{=} \frac{1}{2}|I|^{-1/2} W_I^{-1}(W_{I_1} - W_{I_2})(W_I)^{-1/2} \tag{5.1}$$

The constructed functions $g_I^k = f_I^k + \chi_I \tilde{\boldsymbol{e}}_I^k$ are orthogonal to each other, but not necessarily normalized. It is possible to show that the system is almost normalized, that means

$$0 < \inf_{I,k} \|g_I^k\|_W \leq \sup_{I,k} \|g_I^k\|_W < \infty$$

We shall prove only that $\sup_{I,k} \|g_I^k\|_W < \infty$, since we need only this inequality for our purposes. It is sufficient to show that the vectors $\tilde{\boldsymbol{e}}_I^k$ we added can not be too big, namely that

$$\int_I (W(t)\tilde{\boldsymbol{e}}_I^k, \tilde{\boldsymbol{e}}_I^k)dt = (W(I)A_I\boldsymbol{e}_I^k, A_I\boldsymbol{e}_I^k) \le C < \infty .$$

or, recalling the formula for $A_I$,

$$\frac{1}{2}\left\|(W_I)^{-1/2}\,(W_{I_1} - W_{I_2})\,(W_I)^{-1/2}\boldsymbol{e}_I^k\right\| \le C$$

But operators $(W_I)^{-1/2}\,(W_{I_1} - W_{I_2})\,(W_I)^{-1/2}$ are uniformly bounded. It follows, for example, from Theorem 4.1, because the norms of these operators forms a Carleson measure and therefore they have to be uniformly bounded.

There is also a simple direct proof of this fact. First of all $W_{I_1} \le 2W_I$ since clearly $W(I_1) \le W(I)$. Therefore

$$(W_I)^{-1/2}W_{I_1}(W_I)^{-1/2} \le 2(W_I)^{-1/2}W_I(W_I)^{-1/2} = 2\boldsymbol{I}$$

and therefore $\|(W_I)^{-1/2}W_{I_1}(W_I)^{-1/2}\| \le 2$. Repeat the same for $I_2$ and we are done.

Now we are going to reduce the theorem to some imbedding theorem, kind of weighted Carleson imbedding theorem. First of all, since $g_I^k$ are orthogonal in $L^2(W)$

$$\sum_{\substack{I\in\mathcal{D},\\ k=1,2,\dots,d}} \frac{1}{\|g_I^k\|_W^2}\left|(f, g_I^k)\right|^2 \le \|f\|_W^2\,, \qquad \forall f \in L^2(W)\,,$$

and because $\|g_I^k\|_W$ are uniformly bounded

$$\sum_{\substack{I\in\mathcal{D},\\ k=1,2,\dots,d}} \left|(f, g_I^k)\right|^2 \le C\|f\|_W^2\,, \qquad \forall f \in L^2(W)\,. \tag{5.2}$$

We want to prove the first assertion of Theorem 5.1 for the system of subspaces $E_I$, or equivalently, since the vectors $f_I^k$, $k = 1, 2, \dots, d$ form an orthonormal (in $L^2(W)$) basis in $E_I$,

$$\sum_{\substack{I\in\mathcal{D},\\ k=1,2,\dots,d}} \left|(f, f_I^k)_W\right|^2 \le C\|f\|_W^2\,, \qquad \forall f \in L^2(W)\,. \tag{5.3}$$

Since $g_I^k = f_I^k + A_I\boldsymbol{e}_I^k$ and (5.2) holds, it is sufficient to prove that

$$\sum_{\substack{I\in\mathcal{D},\\ k=1,2,\dots,d}} \left|(f, \chi_I A_I\boldsymbol{e}_I^k)_W\right|^2 \le C\|f\|_W^2\,, \qquad \forall f \in L^2(W)\,. \tag{5.4}$$

Recalling that the system $\boldsymbol{e}_I^k$ $k = 1, 2, \dots, d$ is an orthonormal basis in $\mathbb{C}^d$ (eigenvectors of a selfadjoint matrix $W_I$) and using the notation $f_I = |I|^{-1}\int_I f(t)dt$ we can rewrite the above inequality as

$$\sum_{I\in\mathcal{D}} |I|^2\,\|A_I^*(Wf)_I\|^2 \le C\|f\|_W^2\,, \qquad \forall f \in L^2(W)\,,$$

or equivalently

$$\sum_{I\in\mathcal{D}} |I|^2 \left\| A_I^*(W^{1/2}f)_I \right\|^2 \le C\|f\|^2, \qquad \forall f\in L^2; \tag{5.5}$$

here we have used the fact the multiplication by $W^{-1/2}$ is a unitary operator from $L^2$ to $L^2(W)$.

Let $\mu_I \stackrel{\text{def}}{=} |I|\cdot\|(W_I)^{-1/2}(W_{I_1}-W_{I_2})(W_I)^{-1/2}\|^2$ (note that $\mu_I$ here are different from $\mu_I$ in the proof of Theorem 4.1). Since for any vector $\boldsymbol{f}\in\mathbb{C}^d$ the inequality $\|A_I^*\boldsymbol{f}\| \le \frac{1}{2}(\mu_I)^{1/2}|I|^{-1}\left\|(W_I)^{-1/2}\boldsymbol{f}\right\|$ holds, the inequality (5.5) follows immediately from the following one

$$\sum_{I\in\mathcal{D}} \mu_I \left\|(W_I)^{-1/2}(W^{1/2}f)_I\right\|^2 \le C\|f\|^2, \qquad \forall f\in L^2. \tag{5.6}$$

So we have reduced the theorem to the imbedding (5.6). $\square$

## 6. Proof of the imbedding theorem.

**Theorem 6.1.** *Let a $d\times d$ matrix weight $W$ satisfy the Muckenhoupt condition ($\boldsymbol{A}_2$), and let $\mu_I \stackrel{\text{def}}{=} |I|\cdot\|(W_I)^{-1/2}(W_{I_1}-W_{I_2})(W_I)^{-1/2}\|^2$. Then*

$$\sum_{I\in\mathcal{D}} \mu_I \left\|(W_I)^{-1/2}(W^{1/2}f)_I\right\|^2 \le C\|f\|^2, \qquad \forall f\in L^2. \tag{6.1}$$

*where the constant $C$ depends only on the Muckenhoupt constant of $W$.*

Let us remind the following "dyadic" version of a Carleson imbedding theorem (as it was shown above in Theorem 4.1 the measure $\sum_{I\in\mathcal{D}}\mu_I\delta_{\lambda(I)}$ is Carleson). Let us remind that $\lambda(I)$ is a point in $\mathbb{C}_+$ such that $\operatorname{Im}\lambda(I)=|I|$ and $\operatorname{Re}\lambda(I)$ is the center of $I$.

**Theorem 6.2.** *Let numbers $\mu(I)$, $I\in\mathcal{D}$ be such that the measure $M=\sum_{I\in\mathcal{D}}\mu_I\delta_{\lambda(I)}$ is Carleson. That means there exists a constant $C$ such that that for any dyadic interval $J$*

$$\sum_{I\subset J,\, I\in\mathcal{D}} \mu(I) \le C|J|.$$

*Then for any (scalar) function $f\in L^2$*

$$\sum_{I\in\mathcal{D}} |f_I|^2\mu(I) \le K(C)\cdot\|f\|_2^2.$$

This theorem follows immediately from the classical Carleson imbedding theorem, because for a positive function $f$ the average $f_I$ is estimated as $f_I\le Cf(\lambda(I))$, where $f(\lambda(I))$ is a harmonic extension of $f$ at the point $\lambda(I)$ and $C$ is an absolute constant.

**Proof of Theorem 6.1.** The proof resembles the proof of the Carleson imbedding theorem that can be found in [7].

Denote by $\mathcal{J}$ the imbedding operator, $\mathcal{J}: L^2(\mathbb{C}^d)\to\ell^2(\{\mu(I)\})=\ell^2(\{\mu(I)\},\mathbb{C}^d)$,

$$\mathcal{J}f = \left\{(W_I)^{-1/2}(W^{1/2}f)_I\right\}_{I\in\mathcal{D}} \in \ell^2(\{\mu(I)\}_{I\in\mathcal{D}}).$$

The formal adjoint $\mathcal{J}^*$ to $\mathcal{J}$ is given by the formula

$$\mathcal{J}^* \{\boldsymbol{\alpha}_I\}_{I\in\mathcal{D}} = \sum_{I\in\mathcal{D}} \frac{1}{|I|}\chi_I W^{1/2}(W_I)^{-1/2}\boldsymbol{\alpha}_I\,, \qquad \{\boldsymbol{\alpha}_I\}_{I\in\mathcal{D}} \in \ell^2(\{\mu(I)\},\mathbb{C}^d)\,.$$

A block $(\mathcal{J}\mathcal{J}^*)_{J,I}$ of the operator $(\mathcal{J}\mathcal{J}^*)$ is nonzero only if $J\cap I\neq\varnothing$. Clearly $(\mathcal{J}\mathcal{J}^*)_{I,J} = ((\mathcal{J}\mathcal{J}^*)_{J,I})^*$ and if $I\subset J$

$$(\mathcal{J}\mathcal{J}^*)_{J,I} = \frac{1}{|J|}(W_J)^{-1/2}(W_I)^{1/2}\,.$$

One can estimate for $I\subset J$

$$\|(\mathcal{J}\mathcal{J}^*)_{I,J}\| = \|(\mathcal{J}\mathcal{J}^*)_{J,I}\| = \frac{1}{|J|}\left\|(W_J)^{-1/2}(W_I)^{1/2}\right\| \le \frac{1}{|J|}\left\|\left((W^{-1})_J\right)^{1/2}(W_I)^{1/2}\right\| \stackrel{\text{def}}{=} t_{J,I}\,,$$

and put $t_{I,J}=t_{J,I}$. Put $t_{I,J}=0$ for $I\cap J=\varnothing$. Now to prove the theorem it is sufficient to prove that the matrix $\{t_{I,J}\}_{I,J\in\mathcal{D}}$ generates a bounded operator on $\ell^2(\{\mu(I)\})$.

We shall use the following well known test of boundedness of an integral operator, see [7].

**Theorem 6.3 (Senichkin – Vinogradov Test).** *Let $\mathcal{Z}$ be a measure space, $k$ be a locally summable nonnegative measurable function on $\mathcal{Z}\times\mathcal{Z}$. Let*

$$\int_{\mathcal{Z}} k(s,t)k(s,x)ds \le C\left[k(x,t)+k(t,x)\right]\,, \qquad \text{a.e. on } \mathcal{Z}\times\mathcal{Z}\,.$$

*Then for any positive $g\in L^2(\mathcal{Z})$*

$$\iint_{\mathcal{Z}\times\mathcal{Z}} k(s,t)g(s)g(t)dsdt \le 2C\|g\|^2_{L^2(\mathcal{Z})}\,,$$

*and the integral operator with kernel $k(s,t)$ is bounded in $L^2(\mathcal{Z})$.*

Since the matrix $\{t_{I,J}\}_{I,J\in\mathcal{D}}$ is symmetric, it is sufficient to show that the matrix $\{T_{I,J}\}_{I,J\in\mathcal{D}}$

$$T_{I,J} = \begin{cases} t_{I,J}, & I\subset J \\ 0 & I\not\subset J. \end{cases}$$

generates a bounded operator in $\ell^2(\{\mu(I)\})$.

Since $T_{I,J}\neq 0$ only if $I\subset J$ we need to estimate the sum

$$\sum_{I\in\mathcal{D}} T_{I,J}T_{I,J'}\mu(I)$$

only when $J\cap J'\neq\varnothing$. Without loss of generality we can assume $J'\subset J$, then

$$\sum_{I\in\mathcal{D}} T_{I,J}T_{I,J'}\mu(I) = \sum_{I\in\mathcal{D},\,I\subset J'} T_{I,J}T_{I,J'}\mu(I) =$$
$$= \sum_{I\in\mathcal{D},\,I\subset J'} \frac{1}{|J||J'|}\left\|(W_J)^{-1/2}(W_I)^{1/2}\right\|\left\|(W_{J'})^{-1/2}(W_I)^{1/2}\right\|\mu(I)$$

We can estimate

$$\begin{aligned}\left\|(W_J)^{-1/2}(W_I)^{1/2}\right\| &= \left\|(W_J)^{-1/2}(W_J')^{1/2}(W_J')^{-1/2}(W_I)^{1/2}\right\| \leq \\ &\leq \left\|(W_J)^{-1/2}(W_J')^{1/2}\right\| \left\|(W_J')^{-1/2}(W_I)^{1/2}\right\| .\end{aligned}$$

Hence

$$\sum_{I\in\mathcal{D}} T_{I,J}T_{I,J'}\mu(I) \leq \sum_{I\in\mathcal{D},\, I\subset J'} \frac{1}{|J||J'|} \left\|(W_J)^{-1/2}(W_{J'})^{1/2}\right\| \left\|(W_{J'})^{-1/2}(W_I)^{1/2}\right\|^2 \mu(I)$$

By Lemma 3.5 the weight $(W_{J'})^{-1/2}W(W_{J'})^{-1/2}$ is Muckenhoupt with the same constant as $W$, so we can use Lemma 3.6 to estimate $\left\|(W_{J'})^{-1/2}(W_I)^{1/2}\right\|^2$:

$$\begin{aligned}\left\|(W_{J'})^{-1/2}(W_I)^{1/2}\right\|^2 = \left\|(W_{J'})^{-1/2}W_I(W_{J'})^{-1/2}\right\| \leq \frac{1}{|I|}\int_I \left\|(W_{J'})^{-1/2}W(W_{J'})^{-1/2}\right\| \leq \\ \leq C\left(\frac{1}{|I|}\int_I \left\|(W_{J'})^{-1/2}W^{1/2}\right\|\right)^2 = C\left(\frac{1}{|I|}\int_I \left\|(W_{J'})^{-1/2}W^{1/2}\right\|\chi_{J'}\right)^2\end{aligned}$$

Since the measure $\sum_{I\in\mathcal{D}}\mu(I)\delta_{\lambda(I)}$ is Carleson (see Theorem 4.1), one can use Theorem 6.2

$$\begin{aligned}\frac{|J||J'|}{\|(W_J)^{-1/2}(W_{J'})^{1/2}\|} \sum_{I\in\mathcal{D},\, I\subset J'} T_{I,J}T_{I,J'}\mu(I) \leq C \sum_{I\in\mathcal{D},\, I\subset J'} \left(\frac{1}{|I|}\int_I \left\|(W_{J'})^{-1/2}W^{1/2}\right\|\chi_{J'}\right)^2 \mu(I) \leq \\ \leq C' \left\| \left\|(W_{J'})^{-1/2}W^{1/2}\right\|\chi_{J'}\right\|_{L^2}^2 = C'\int_{J'} \left\|(W_{J'})^{-1/2}W(W_{J'})^{-1/2}\right\| \leq \\ \leq C''\left\|(W_{J'})^{-1/2}\left[\int_{J'} W\right](W_{J'})^{-1/2}\right\| = C''|J'|\,;\end{aligned}$$

the last inequality here holds by Lemma 3.1. Therefore

$$\sum_{I\in\mathcal{D}} T_{I,J}T_{I,J'}\mu(I) \leq \frac{C''}{|J|}\left\|(W_J)^{-1/2}(W_{J'})^{1/2}\right\| = C''(T_{J',J}+T_{J,J'})$$

and we are done. □

## 7. Boundedness of the Hilbert transform.

Let $T$ denote Hilbert transform

$$Tf(x) = \text{p.v.}\,\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{x-t}dt\,.$$

In this section we are going to prove the following theorem.

**Theorem 7.1.** *The operator $T$ is bounded in the weighted space $L^2(W)$ (or, equivalently, that $W^{1/2}TW^{-1/2}$ is bounded in the regular vector-valued space $L^2$) if and only if the matrix weight $W$ satisfies the Muckenhoupt condition ($\boldsymbol{A}_2$).*

**Remark.** The "only if" part of the theorem is already proved, see Lemma 2.2. As one can see from the proof below, the "if" part holds not only for Hilbert Transform, but for more general singular integral operators. In fact it is proved below that any operator whose matrix in the Haar system is estimated by the sum of matrices from Lemmas 7.7, 7.8 is bounded in the weighted space $L^2(W)$ if the weight $W$ satisfies the Muckenhoupt ($\boldsymbol{A}_2$) condition.

**Proof of Theorem 7.1: The beginning.**

To prove the theorem consider the matrix of the operator $W^{1/2}TW^{-1/2}$ with respect to the systems of vectors $W^{1/2}\boldsymbol{f}_I^k$ and $W^{-1/2}\boldsymbol{g}_I^k$, where $\boldsymbol{f}_I^k$ is the vector Haar system constructed in Section 5,

$$\boldsymbol{f}_I^k = w_I^k h_I \boldsymbol{e}_I^k\,, \qquad w_I^k = \left\| [W_I]^{-1/2}\boldsymbol{e}_I^k \right\|;$$

here $h_I$ is usual (scalar) Haar function on a dyadic interval $I$, and vectors $\boldsymbol{e}_I^1, \boldsymbol{e}_I^2, \dots, \boldsymbol{e}_I^d \in \mathbb{C}^d$ are eigenvectors of $W_I$. The system $\boldsymbol{g}_I^k$ is the corresponding system for $W^{-1}$ namely

$$\boldsymbol{g}_I^k = \tilde{w}_I^k h_I \tilde{\boldsymbol{e}}_I^k\,, \qquad \tilde{w}_I^k = \left\| [(W^{-1})_I]^{-1/2}\tilde{\boldsymbol{e}}_I^k \right\|,$$

where $\tilde{\boldsymbol{e}}_I^1, \tilde{\boldsymbol{e}}_I^2, \dots, \tilde{\boldsymbol{e}}_I^d \in \mathbb{C}^d$ are eigenvectors of $(W^{-1})_I$.

By Theorem 5.2 systems $\boldsymbol{f}_I^k$ and $\boldsymbol{g}_I^k$ form Riesz bases in $L^2(W)$ and $L^2(W^{-1})$ respectively, so the systems $W^{1/2}\boldsymbol{f}_I^k$ and $W^{-1/2}\boldsymbol{g}_I^k$ are two Riesz bases in $L^2$. Therefore it is sufficient to show that the matrix

$$\left\{ (W^{1/2}TW^{-1/2} \cdot W^{1/2}\boldsymbol{f}_J^l, W^{-1/2}\boldsymbol{g}_I^k) \right\}_{I,J\in\mathcal{D},\, k,l=1,2,\dots,d}$$

is a matrix of a bounded operator in $\ell^2$. One can rewrite this matrix as

$$\left\{ w_J^l \tilde{w}_I^k (Th_J, h_I)(\boldsymbol{e}_J^l, \tilde{\boldsymbol{e}}_I^k) \right\}_{I,J\in\mathcal{D},\, k,l=1,2,\dots,d}\,,$$

or, gathering $d\times d$ blocks and recalling the definitions of $w_I^k$ and $\tilde{w}_J^l$,

$$\left\{ (Th_J, h_I)[(W^{-1})_I]^{-1/2}[W_J]^{-1/2} \right\}_{I,J\in\mathcal{D}}\,.$$

If we estimate the norm of each block and the resulting matrix of positive number turns out to be the matrix of a bounded operator in $\ell^2$, we are done.

By Corollary 3.3

$$\left\| [(W^{-1})_I]^{-1/2}[W_J]^{-1/2} \right\| \le \left\| [W_I]^{1/2}[(W^{-1})_J]^{1/2} \right\|.$$

The number $(Th_J, h_I)$ is also easy to estimate. For a dyadic interval $J$ let $Z(J)$ denote the interval $3J$ (the interval $J$ itself and the two neighbors of the same size), and $s(J)$ denotes three-point set formed of the center point and endpoints of $J$ ($Th_J$ has logarithmic singularities at points of $s(J)$, that explains the notation). Recall that $\lambda(I)$ denotes the point in $\mathbb{C}_+$ such that $\operatorname{Re}\lambda(I)$ is the center of $I$ and $\operatorname{Im}\lambda(I) = |I|$. If $|I| \le |J|$ we denote by $\rho_{I,J}$ the distance from $\lambda(I)$ to $s(J)$.

**Lemma 7.2.** *If $|I| \le |J|$ then*

$$|(Th_I, h_J)| = |(Th_J, h_I)| \le \begin{cases} C\frac{|I|^{3/2}|J|^{3/2}}{\rho^3_{I,J}}, & \text{if } I \not\subset Z(J) \\ C\frac{|I|^{3/2}}{\rho_{I,J}}\frac{1}{|J|^{1/2}}, & \text{if } I \subset Z(J) \end{cases}$$

This is a standard and explicit computation and we omit the proof.

## 7.1. Some auxiliary matrices.

We are going to estimate our matrix by the sum of two matrices: one deals with the singularity at $\infty$, another one with the singularity that occurs when $I$ is close to $s(J)$.

We need the following two tests of the boundedness of an integral operator.

**Theorem 7.3 (Modified Senichkin — Vinogradov test, 1).** *Let $\mathcal{Z}$ be a measure space, $k_1$, $k_2$ be locally summable nonnegative measurable functions on $\mathcal{Z} \times \mathcal{Z}$, and let $K = k_1 + k_2$. Suppose that*

$$\int_{\mathcal{Z}} k_{1,2}(s,t)k_{1,2}(s,x)ds \le C\left[K(x,t) + K(t,x)\right], \qquad \text{a.e. on } \mathcal{Z}\times\mathcal{Z}\,.$$

*Then the integral operators with kernels $k_1$ and $k_2$ are bounded on $L^2$*

**Proof.** Let $f \in L^2$, $\|f\|_2 = 1$ be a nonnegative function such that

$$P \stackrel{\text{def}}{=} \iint_{\mathcal{Z}\times\mathcal{Z}} k_1(s,t)f(s)f(t)dsdt < \infty\,, \qquad Q \stackrel{\text{def}}{=} \iint_{\mathcal{Z}\times\mathcal{Z}} k_2(s,t)f(s)f(t)dsdt < \infty\,.$$

Then

$$\begin{aligned} P^2 = \left(\int_{\mathcal{Z}} f(s)\left(\int_{\mathcal{Z}} k_1(s,t)f(t)dt\right)ds\right)^2 &\le \|f\|^2_{L^2(\mathcal{Z})}\cdot\left\|\int_{\mathcal{Z}} k_1(.,t)f(t)dt\right\|^2 \le \\ &\le \int_{\mathcal{Z}} ds\left(\iint_{\mathcal{Z}\times\mathcal{Z}} k_1(s,t)k_1(s,x)f(t)f(x)dtdx\right) = \\ &= \iint_{\mathcal{Z}\times\mathcal{Z}} f(t)f(x)\left(\int_{\mathcal{Z}} k_1(s,t)k_1(s,x)ds\right) \le \\ &\le 2C\int_{\mathcal{Z}}[k_1(t,x)+k_2(t,x)]f(t)f(x)dtdx = 2C(P+Q)\,, \end{aligned}$$

and similarly,

$$Q^2 \le 2C(P+Q)\,.$$

Therefore

$$(P+Q)^2 \le 2(P^2+Q^2) \le 8C(P+Q)\,.$$

This implies $(P+Q) \le 8C$, so $P \le 8C$, $Q \le 8C$. From here one can conclude that the corresponding integral operators are bounded and their norms can be estimated above as $2\sqrt{2C}$. □

**Theorem 7.4 (Modified Senichkin — Vinogradov test, 2).** *Let $\mathcal{Z}$ be a measure space, $k$, $b$ be locally summable nonnegative measurable functions on $\mathcal{Z}\times\mathcal{Z}$, and the integral operator with kernel $b$ is bounded (with the norm $B$). Suppose that*

$$\int_{\mathcal{Z}} k(s,t)k(s,x)ds \le C\left[k(x,t)+k(t,x)\right]+b(x,t)\,, \qquad \text{a.e. on } \mathcal{Z}\times\mathcal{Z}\,.$$

*Then the integral operator with kernel $k$ is bounded on $L^2(\mathcal{Z})$ (with norm estimated by a constant depending on $C$ and $B$).*

**Proof.** As in the proof of Theorem 7.3 take $f\in L^2$, $\|f\|_2=1$ such that

$$Q \stackrel{\text{def}}{=} \iint_{\mathcal{Z}\times\mathcal{Z}} k(s,t)f(s)f(t)dsdt < \infty\,.$$

Applying the same reasoning as in Theorem 7.3 one can conclude that $Q^2\le 2CQ+B$. $\square$

Let $w$ be a (scalar) Muckenhoupt weight, $I$ be an interval and $\lambda(I)\in\mathbb{C}_+$ be a point such that $\operatorname{Re}\lambda(I)$ is a center of $I$ and $\operatorname{Im}\lambda(I)=|I|$. Let $w(\lambda)$ be the Poisson extension of $w$ at the point $\lambda\in\mathbb{C}_+$.

**Lemma 7.5.** *Let $w$ be a scalar Muckenhoupt weight. There exists a constant $C$ depending only on Muckenhoupt constant of $w$ such that $w(\lambda(I))\le Cw_I$.*

**Remark.** The opposite inequality (with some absolute constant) is trivial.

**Corollary 7.6.** *For a $d\times d$ matrix Muckenhoupt weight $W$ there exists a constant $C$ depending only on the Muckenhoupt constant of $W$ such that $W(\lambda(I))\le CW_I$.*

**Proof.** Take a vector $\boldsymbol{e}\in\mathbb{C}^d$. The scalar weight $w$, $w(t)=(W(t)\boldsymbol{e},\boldsymbol{e})$ is clearly a Muckenhoupt weight with the same or better constant as $W$. So by Lemma 7.5 $(W(\lambda(I))\boldsymbol{e},\boldsymbol{e})\le C(W_I\boldsymbol{e},\boldsymbol{e})$. $\square$

Let us recall that $\mathcal{D}$ denotes the set of all dyadic intervals. Denote by $\{t_{I,J}\}_{I,J\in\mathcal{D}}$ a symmetric matrix defined by

$$t_{I,J}=\frac{|I|^{3/2}}{|J|^{3/2}}\cdot\frac{|J|^2}{\left|\lambda(I)-\overline{\lambda(J)}\right|^2}\,, \qquad |I|\le|J| \tag{7.2}$$

**Lemma 7.7.** *Given a $d\times d$ matrix Muckenhoupt weight $W$ the matrix*

$$\left\{t_{I,J}\left\|[(W^{-1})_J]^{1/2}[W_I]^{1/2}\right\|\right\}_{I,J\in\mathcal{D}}$$

*generates a bounded operator in $\ell^2$ with the norm at most $M$ where $M$ depends only on $d$ and the Muckenhoupt constant of $W$.*

**Proof.** Since the matrix $t_{I,J}$ is symmetric, it is sufficient to show that matrices $T^1_{I,J}$, $T^2_{I,J}$ defined by $T^1_{I,J} = T^2_{I,J} = 0$ for $|J| < |I|$ and

$$T^1_{I,J} = t_{I,J} \left\| [(W^{-1})_J]^{1/2} [W_I]^{1/2} \right\| , \qquad T^2_{I,J} = t_{I,J} \left\| [W_J]^{1/2} [(W^{-1})_I]^{1/2} \right\| , \qquad |I| \le |J|$$

generates bounded operators. Let us apply the modified Senichkin — Vinogradov Test (Theorem 7.3). Take some interval $J'$. Since in Theorem 7.3 we need an estimate by a symmetric sum $K(x,t) + K(t,x)$, we can assume without loss of generality that $|J'| \le |J|$. We want to estimate

$$\sum_{I:\ |I| \le |J'|} T^{1,2}_{I,J} T^{1,2}_{I,J'}$$

First of all note that for all $s, x, t \in \mathbb{C}_+$

$$|x - \bar{t}| = |x - \bar{s} + \bar{s} - \bar{t}| \le |x - \bar{s}| + |s - t| \le |x - \bar{s}| + |s - \bar{t}|$$

and therefore

$$|x - \bar{s}|^{-1} |s - \bar{t}|^{-1} \le |x - \bar{t}|^{-1} \left( |x - \bar{s}|^{-1} + |s - \bar{t}|^{-1} \right) .$$

Hence, if $|I| \le |J'| \le |J|$ then

$$t_{I,J} t_{I,J'} = \frac{|I|^{3/2}}{|J|^{3/2}} \cdot \frac{|I|^{3/2}}{|J'|^{3/2}} \cdot \frac{|J|^2}{\left|\lambda(I) - \overline{\lambda(J)}\right|^2} \cdot \frac{|J'|^2}{\left|\lambda(I) - \overline{\lambda(J')}\right|^2} \le$$

$$\le \frac{|J'|^{1/2}}{|J|^{3/2}} \cdot |J|^2 |I|^3 \frac{1}{\left|\lambda(J') - \overline{\lambda(J)}\right|^2} \left( \frac{1}{\left|\lambda(I) - \overline{\lambda(J)}\right|^2} + \frac{1}{\left|\lambda(I) - \overline{\lambda(J')}\right|^2} \right) =$$

$$= t_{J',J} |I|^3 |J'|^{-1} \left( \frac{1}{\left|\lambda(I) - \overline{\lambda(J)}\right|^2} + \frac{1}{\left|\lambda(I) - \overline{\lambda(J')}\right|^2} \right)$$

Let us note that

$$\left\| [(W^{-1})_J]^{1/2} [W_I]^{1/2} \right\| = \left\| [(W^{-1})_J]^{1/2} [W_{J'}]^{1/2} [W_{J'}]^{-1/2} [W_I]^{1/2} \right\| \le$$

$$\le \left\| [(W^{-1})_J]^{1/2} [W_{J'}]^{1/2} \right\| \left\| [W_{J'}]^{-1/2} [W_I]^{1/2} \right\| \le$$

$$\le \left\| [(W^{-1})_J]^{1/2} [W_{J'}]^{1/2} \right\| \left\| [(W^{-1})_{J'}]^{1/2} [W_I]^{1/2} \right\|$$

(the last inequality holds by Corollary 3.3). Therefore

$$\sum_{I:\ |I| \le |J'|} \left\| [(W^{-1})_J]^{1/2} [W_I]^{1/2} \right\| \left\| [(W^{-1})_{J'}]^{1/2} [W_I]^{1/2} \right\| \frac{|I|^3}{\left|\lambda(I) - \overline{\lambda(J')}\right|^2} \le$$

$$\le \left\| [(W^{-1})_J]^{1/2} [W_{J'}]^{1/2} \right\| \sum_{I:\ |I| \le |J'|} \left\| [(W^{-1})_{J'}]^{1/2} [W_I]^{1/2} \right\|^2 |I|^2 \frac{|J'|}{\left|\lambda(I) - \overline{\lambda(J')}\right|^2} \le$$

$$C \left\| [(W^{-1})_J]^{1/2} [W_{J'}]^{1/2} \right\| \sum_{k=0}^{\infty} |J'| 2^{-k} \int_{\mathbb{R}} \frac{|J'|}{\left|t - \overline{\lambda(J')}\right|^2} \left\| [(W^{-1})_{J'}]^{1/2} W(t) [(W^{-1})_{J'}]^{1/2} \right\| dt \le$$

$$\leq C_1 \left\| [(W^{-1})_J]^{1/2} [W_{J'}]^{1/2} \right\| |J'| \left\| \int_{\mathbb{R}} \frac{|J'|}{\left| t - \overline{\lambda(J')} \right|^2} [(W^{-1})_{J'}]^{1/2} W(t) [(W^{-1})_{J'}]^{1/2} dt \right\|$$

(the last inequality holds by Lemma 3.1.) By Lemma 3.5 the weight

$$[(W^{-1})_{J'}]^{1/2} W(.) [(W^{-1})_{J'}]^{1/2}$$

is a Muckenhoupt weight with the same or better constant than $W$, therefore by Corollary 7.6

$$\int_{\mathbb{R}} \frac{|J'|}{\left| t - \overline{\lambda(J')} \right|^2} [(W^{-1})_{J'}]^{1/2} W(t) [(W^{-1})_{J'}]^{1/2} dt \leq A [(W^{-1})_{J'}]^{1/2} W(\lambda(J')) [(W^{-1})_{J'}]^{1/2} \leq$$

$$\leq C [(W^{-1})_{J'}]^{1/2} W_{J'} [(W^{-1})_{J'}]^{1/2} \leq C' \boldsymbol{I}$$

(the Muckenhoupt condition). Therefore we can continue the chain of inequalities:

$$\sum_{I:\ |I| \leq |J'|} \left\| [(W^{-1})_J]^{1/2} [W_I]^{1/2} \right\| \left\| [(W^{-1})_{J'}]^{1/2} [W_I]^{1/2} \right\| \frac{|I|^3}{\left| \lambda(I) - \overline{\lambda(J')} \right|^2} \leq$$

$$\leq C \left\| [(W^{-1})_J]^{1/2} [W_{J'}]^{1/2} \right\| |J'| .$$

Similarly,

$$\left\| [(W^{-1})_{J'}]^{1/2} [W_I]^{1/2} \right\| = \left\| [(W^{-1})_{J'}]^{1/2} [W_J]^{1/2} [W_J]^{-1/2} [W_I]^{1/2} \right\| \leq$$

$$\leq \left\| [(W^{-1})_{J'}]^{1/2} [W_J]^{1/2} \right\| \left\| [W_J]^{-1/2} [W_I]^{1/2} \right\| \leq$$

$$\leq \left\| [(W^{-1})_{J'}]^{1/2} [W_J]^{1/2} \right\| \left\| [(W^{-1})_J]^{1/2} [W_I]^{1/2} \right\| ,$$

and therefore

$$\sum_{I:\ |I| \leq |J'|} \left\| [(W^{-1})_J]^{1/2} [W_I]^{1/2} \right\| \left\| [(W^{-1})_{J'}]^{1/2} [W_I]^{1/2} \right\| \frac{|I|^3}{\left| \lambda(I) - \overline{\lambda(J)} \right|^2} \leq$$

$$\leq \left\| [(W^{-1})_{J'}]^{1/2} [W_J]^{1/2} \right\| \sum_{I:\ |I| \leq |J'|} \left\| [(W^{-1})_J]^{1/2} [W_I]^{1/2} \right\|^2 |I|^2 \frac{|J|}{\left| \lambda(I) - \overline{\lambda(J)} \right|^2} \leq$$

$$\leq C \left\| [(W^{-1})_{J'}]^{1/2} [W_J]^{1/2} \right\| \sum_{k=0}^{\infty} |J'| 2^{-k} \int_{\mathbb{R}} \frac{|J|}{\left| t - \overline{\lambda(J)} \right|^2} \left\| [(W^{-1})_J]^{1/2} W(t) [(W^{-1})_J]^{1/2} \right\| dt \leq$$

$$\leq C_1 \left\| [(W^{-1})_{J'}]^{1/2} [W_J]^{1/2} \right\| |J'| \left\| \int_{\mathbb{R}} \frac{|J|}{\left| t - \overline{\lambda(J)} \right|^2} [(W^{-1})_J]^{1/2} W(t) [(W^{-1})_J]^{1/2} dt \right\| \leq$$

$$\leq C \left\| [(W^{-1})_{J'}]^{1/2} [W_J]^{1/2} \right\| |J'| .$$

Gathering all the calculations together we obtain (assuming $|J'| \leq |J|$ and therefore $|I| \leq |J'|$)

$$\sum_I T^1_{I,J} T^1_{I,J'} \leq C t_{J',J} \left( \left\| [(W^{-1})_J]^{1/2} [W_{J'}]^{1/2} \right\| + \left\| [(W^{-1})_{J'}]^{1/2} [W_J]^{1/2} \right\| \right) = C (T^1_{J',J} + T^2_{J',J}) .$$

The weight $W^{-1}$ is a Muckenhoupt weight, and the similar reasoning implies

$$\sum_I T^2_{I,J} T^2_{I,J'} \le C(T^1_{J',J} + T^2_{J',J}) .$$

Therefore by Theorem 7.3 both matrices generate bounded operators. ☐

Denote by $\{k_{I,J}\}_{I,J\in\mathcal{D}}$ a symmetric matrix defined for $|I| \le |J|$ by

$$k_{I,J} = \begin{cases} \frac{|I|^{3/2}}{\rho_{I,J}} \frac{1}{|J|^{1/2}}, & I \subset Z(J) \\ 0, & I \not\subset Z(J) \end{cases} \tag{7.3}$$

**Lemma 7.8.** *Given a matrix Muckenhoupt weight $W$ the matrix*

$$\left\{k_{I,J} \| [W_I]^{1/2} [(W^{-1})_J]^{1/2} \| \right\}_{I,J\in\mathcal{D}}$$

*is the matrix of a bounded in $\ell^2$ operator (with norm depending on the Muckenhoupt constant of $W$).*

**Proof of Theorem 7.1: the completion.** Lemmas 7.7 and 7.8 imply immediately that the Hilbert transform $T$ is bounded in $L^2(W)$. Indeed, one can see from Lemma 7.2 that $|(Th_I, h_J)| \| [W_I]^{1/2} [(W^{-1})_J]^{1/2} \|$ is estimated above by

$$C \cdot t_{I,J} \| [W_I]^{1/2} [(W^{-1})_J]^{1/2} \|$$

if $I \not\subset Z(J)$ and $J \not\subset Z(I)$, and by

$$C \cdot k_{I,J} \| [W_I]^{1/2} [(W^{-1})_J]^{1/2} \|$$

if $I \subset Z(J)$ or $J \subset Z(I)$. So it is estimated by the sum of these matrices, and each matrix is bounded in $\ell^2$ by Lemmas 7.7 and 7.8 respectively. ☐

**Proof of Lemma 7.8.** Since everything is symmetric with respect to change of $W$ to $W^{-1}$ it is sufficient to prove that only one "half" of the matrix is a bounded operator in $\ell^2$; the reasoning for the other "half" is absolutely the same. Namely we shall prove that the matrix $K_{I,I}$, $K_{I,J} = 0$ if $|J| < |I|$ and

$$K_{I,J} = k_{I,J} \| [W_I]^{1/2} [(W^{-1})_J]^{1/2} \| \qquad \text{if } |I| \le |J|$$

generates a bounded operator in $\ell^2$.

We apply Theorem 7.4 to the matrix $K_{I,J}$. Suppose that $J'$ is a dyadic interval, $|J'| \le |J|$. First, acting like in the proof of Lemma 7.7 notice that

$$\begin{aligned} \left\| [(W^{-1})_J]^{1/2} [W_I]^{1/2} \right\| &= \left\| [(W^{-1})_J]^{1/2} [W_{J'}]^{1/2} [W_{J'}]^{-1/2} [W_I]^{1/2} \right\| \le \\ &\le \left\| [(W^{-1})_J]^{1/2} [W_{J'}]^{1/2} \right\| \left\| [W_{J'}]^{-1/2} [W_I]^{1/2} \right\| \le \\ &\le \left\| [(W^{-1})_J]^{1/2} [W_{J'}]^{1/2} \right\| \left\| [(W^{-1})_{J'}]^{1/2} [W_I]^{1/2} \right\| \end{aligned}$$

(the last inequality holds by Corollary 3.3). Therefore

$$\sum_{I\in\mathcal{D},\,|I|\le|J'|} K_{I,J}K_{I,J'} \le \left\|[(W^{-1})_J]^{1/2}[W_{J'}]^{1/2}\right\| \sum_{I\in\mathcal{D},\,I\subset Z(J')} k_{I,J}k_{I,J'} \left\|[(W^{-1})_{J'}]^{1/2}[W_I]^{1/2}\right\|^2$$

For a set $A$ denote

$$\omega(A) = \operatorname{trace}\left\{[(W^{-1})_{J'}]^{1/2}[W(A)][(W^{-1})_{J'}]^{1/2}\right\}$$

and let $\omega_A \stackrel{\text{def}}{=} \frac{1}{|A|}\omega(A)$. Clearly

$$\left\|[(W^{-1})_{J'}]^{1/2}[W(A)]^{1/2}\right\|^2 = \left\|[(W^{-1})_{J'}]^{1/2}[W(A)][(W^{-1})_{J'}]^{1/2}\right\| \asymp \omega(A)\,;$$

the inequality $\le$ is trivial, and the inequality $\ge$ follows from Lemma 3.1

We consider 3 cases:

**Case 1:** $J' \subset Z(J)$ and $\rho_{J',J} \ge 10|J'|$. In this case consider dyadic Whitney intervals of $Z(J')\setminus s(J')$, and let $A_J$ denote the union of all such intervals of length $|J'|2^{-j}$ (there are at most 6 such intervals). If a dyadic interval $I\subset A_j$ ($I$ can be of arbitrary length), then $\rho_{I,J'} \asymp |A_j| \asymp |J'|2^{-j}$. Taking into account that under the above assumptions $\rho_{I,J}\asymp\rho_{J',J}$, one can conclude

$$\left\|[(W^{-1})_J]^{1/2}[W_{J'}]^{1/2}\right\|^{-1} \sum_{I\in\mathcal{D},\,|I|\le|J'|} K_{I,J}K_{I,J'} \le C \sum_{I\in\mathcal{D},\,I\subset Z(J')} k_{I,J}k_{I,J'}\omega_I \le$$

$$\le \frac{C}{\rho_{J',J}|J|^{1/2}|J'|^{1/2}} \sum_{I\in\mathcal{D},\,I\subset Z(J')} \frac{|I|^2\omega(I)}{\rho_{I,J'}}$$

Estimating the last sum we have

$$\sum_{I\in\mathcal{D},\,I\subset Z(J')} \frac{|I|^2\omega(I)}{\rho_{I,J'}} \le C\sum_{j=0}^{\infty}\sum_{I\subset A_j} \frac{|I|^2\omega(I)}{\rho_{I,J'}} \le$$

$$\le C\sum_{j=0}^{\infty} \frac{\sum_{t=0}^{\infty}|J'|^2 2^{-2j}2^{-2t}\omega(A_j)}{|A_j|} \le C\sum_{j=0}^{\infty}|J'|2^{-j}\omega(A_j) \le C|J'|\omega(J')$$

The Muckenhoupt condition implies that $\omega(J')\le C|J'|$ and therefore

$$\sum_{I\in\mathcal{D},\,|I|\le|J'|} K_{I,J}K_{I,J'} \le C\left\|[(W^{-1})_J]^{1/2}[W_{J'}]^{1/2}\right\| \frac{|J'|^{3/2}}{\rho_{J',J}|J|^{1/2}} = C(K_{J',J}+K_{J,J'})\,,$$

and we are done in this case.

**Case 2:** $J'\not\subset Z(J)$. In this case $\rho_{J',J}\ge|J|$, and in most situations $K_{I,J}K_{I,J'}=0$. If $K_{I,J}K_{I,J'}\ne0$ for some $I$, then $\rho_{J',J}\le 3|J|$, so $\rho_{J',J}\asymp|J|$. We can apply the same reasoning as in Case 1 to get the estimate

$$\sum_{I\in\mathcal{D},\,|I|\le|J'|} K_{I,J}K_{I,J'} \le C\left\|[(W^{-1})_J]^{1/2}[W_{J'}]^{1/2}\right\| \frac{|J'|^{3/2}}{\rho_{J',J}|J|^{1/2}}\,.$$

The problem is that for our choice of $J$ and $J'$ we have $K_{J',J} + K_{J,J'} = 0$, so we can not estimate the right side by this sum. But then $\frac{|J'|^{3/2}}{\rho_{J',J}|J|^{1/2}} \asymp \frac{|J'|^{3/2}}{|J|^{3/2}} \asymp t_{J',J}$, so

$$\sum_{I\in\mathcal{D},\,|I|\le|J'|} K_{I,J}K_{I,J'} \le CT_{J',J}\,,$$

where $\{T_{I,J}\}_{I,J\in\mathcal{D}}$ is the matrix from Lemma 7.7 which we know generates a bounded operator in $\ell^2$.

**Case 3:** $J' \subset J$, $\rho_{J',J} < 10|J'|$. Note that $\rho_{J',J} \ge |J'|$ always, so $\rho_{J',J} \asymp |J'|$. We sum over all dyadic $I \subset Z(J')$.

Consider dyadic Whitney intervals for $Z(J')\setminus(s(J')\cup s(J))$, and denote by $A_j$ the union of all intervals of length $|J'|2^{-j}$. Note that there exists at most 12 such intervals (2 for each point of $s(J')\cup s(J)$; of course it is not a sharp estimate).

Let $\rho_I \overset{\text{def}}{=} \min\{\rho_{I,J}, \rho_{I,J'}\}$, then $\rho_I \asymp |J'|2^{-j}$ if $I\subset A_j$. Again

$$\sum_{I\in\mathcal{D},\,|I|\le|J'|} K_{I,J}K_{I,J'} \le \left\|[(W^{-1})_J]^{1/2}[W_{J'}]^{1/2}\right\| \sum_{I\in\mathcal{D},\,I\subset Z(J')} k_{I,J}k_{I,J'}\omega_I \le$$

$$\le C\frac{\left\|[(W^{-1})_J]^{1/2}[W_{J'}]^{1/2}\right\|}{|J|^{1/2}|J'|^{1/2}} \sum_{I\in\mathcal{D},\,I\subset Z(J')} \frac{|I|^2\omega(I)}{\rho_I^2}\,.$$

Estimating the last sum we obtain

$$\sum_{I\in\mathcal{D},\,I\subset Z(J')} \frac{|I|^2\omega(I)}{\rho_I^2} \le C\sum_{j=0}^{\infty} \frac{\sum_{t=0}^{\infty}|J'|^2 2^{-2j}2^{-2t}\omega(A_j)}{|A_j|^2} \le C\omega(J') = C|J'|\omega_{J'}\,.$$

Since $\rho_{J',J} \asymp |J'|$ and by the Muckenhoupt condition $\omega_{J'} \le C$,

$$\frac{|J'|^{1/2}}{|J|^{1/2}}\omega_{J'} \le C\frac{|J'|^{3/2}}{\rho_{J',J}|J|^{1/2}}$$

and therefore

$$\sum_{I\in\mathcal{D},\,|I|\le|J'|} K_{I,J}K_{I,J'} \le C\left\|[(W^{-1})_J]^{1/2}[W_{J'}]^{1/2}\right\| \frac{|J'|^{3/2}}{\rho_{J',J}|J|^{1/2}} = C(K_{J',J}+K_{J,J'})\,.$$

$\square$

# 8. Unconditional bases of wavelets.

## 8.1. Construction of a wavelet basis.

Let we have so-called biorthogonal wavelets, i.e. functions $\varphi$, $\psi$ such that the system of dilations and shifts of $\varphi$ is an unconditional basis in $L^2$, and dilations and shifts of $\psi$ forms a dual system to it. More precisely, given a dyadic interval $I$ with left endpoint $x_I$ and length $|I| = 2^{-\nu}$ define

$$\varphi_I(x) = |I|^{-1/2}\varphi(2^\nu(x - x_I))\,, \qquad \psi_I(x) = |I|^{-1/2}\psi(2^\nu(x - x_I))\,.$$

Note that for $I = [0,1]$, $\varphi_I = \varphi$, $\psi_I = \psi$, and $\|\varphi_I\|_2 = \|\varphi\|_2$, $\|\psi_I\|_2 = \|\psi\|_2$.

Formally we only assume that the system of functions $\varphi_I$, $I \in \mathcal{D}$ is a complete minimal (and therefore, uniformly minimal) system in the (non-weighted) scalar $L^2$, and the system $\psi_I$, $I \in \mathcal{D}$ is a dual system to it (of course we can replace $\varphi$ and $\psi$: everything is absolutely symmetric here).

We assume that functions $\varphi$, $\psi$ satisfy some very minimal conditions, namely

1. $\varphi$, $\psi$ are bounded, and moreover $|\varphi(x)| \le C/|x|^2$, $|\psi(x)| \le C/|x|^2$ at $\infty$;

2. $\varphi$, $\psi$, are absolutely continuous and $\varphi'$ and $\psi'$ are bounded and satisfy the estimates $|\varphi'(x)| \le C/|x|^2$, $|\psi'(x)| \le C/|x|^2$ at $\infty$;

Condition 1 implies

3. $\varphi, \psi \in L^1(\mathbb{R})$, and, moreover $\int_R^\infty |\varphi|$, $\int_{-\infty}^{-R} |\varphi|$, $\int_R^\infty |\psi|$, $\int_{-\infty}^{-R} |\psi| \le C/|R|$

Conditions 1 — 3 imply

4. $\int_\mathbb{R} \varphi(t)dt = \int_\mathbb{R} \psi(t)dt = 0$.

Indeed, if, for example, $\int_\mathbb{R} \varphi_I(t)dt \neq 0$, it is an easy exercise to show that $\varphi$ is not orthogonal to $\psi_I$ for some big enough dyadic interval $I$.

Denote $W^I \stackrel{\text{def}}{=} \int W|\varphi|^2$, and let $\boldsymbol{e}_I^k$, $k = 1, 2, \dots, d$ be an orthonormal basis of eigenvalues of $W^I$. Put

$$w_I^k = \left[(W^I \boldsymbol{e}_I^k, \boldsymbol{e}_I^k)\right]^{-1/2} = \left[(\left[W^I\right]^{-1} \boldsymbol{e}_I^k, \boldsymbol{e}_I^k)\right]^{1/2} = \left\|\left[W^I\right]^{-1/2} \boldsymbol{e}_I^k\right\|\,.$$

(note that numbers $w_I^k$ here are different from those in section 5). The system

$$\boldsymbol{f}_I^k = w_I^k \varphi_I \boldsymbol{e}_I^k\,, \qquad I \in \mathcal{D},\ k = 1, 2, ..., d$$

is a normalized ($\|\boldsymbol{f}_I^k\|_{L^2(W)} = 1$) system in $L^2(W)$. Note that we can not apriori say that it is complete in $L^2(W)$.

Define a system of subspaces $E_I = E_I^\varphi \subset L^2(W)$

$$E_I = \mathcal{L}(\boldsymbol{f}_I^k\ :\ k = 1, 2, \dots, d)\,, \qquad I \in \mathcal{D}$$

Note, that the subspaces $E_I$ do not depend on the weight $W$.

We can easily write the spectral projections $\mathcal{P}^I$ defined in section 5.1

$$\mathcal{E}^I f = \begin{cases} f, & f \in E_I \\ 0, & f \in E_J,\ J \neq I \end{cases}$$

Namely, $\mathcal{E}^I f = \varphi \cdot (\int_{\mathbb{R}} \overline{\psi(t)} f(t) dt)$

Acting as in the proof of Lemma 2.1 one can show that the norm $\|\mathcal{P}^I\|$ is exactly

$$\left\| \left[ \int W |\varphi_I|^2 \right]^{1/2} \left[ \int W^{-1} |\psi_I|^2 \right]^{1/2} \right\|$$

Property 1 of $\varphi$, $\psi$ implies that $|\varphi_I|^2$ and $|\psi_I|^2$ is majorized by Poisson kernel $\mathcal{P}_{\lambda(I)}$

$$|\varphi_I(t)|^2\,,\ |\psi_I(t)|^2 \le \frac{C|I|}{|t - \lambda(I)|^2} = C\mathcal{P}_{\lambda(I)} \qquad t \in \mathbb{R}\,,$$

therefore by Corollary 7.6 $\int W|\varphi_I|^2 \le CW(\lambda(I)) \le C'W_I$ and $\int W^{-1}|\psi_I|^2 \le CW^{-1}(\lambda(I)) \le C'(W^{-1})_I$. Hence the Muckenhoupt condition $(\boldsymbol{A}_2)$ implies that the system of subspaces $E_I$ is uniformly minimal.

We want again to bring the reader's attention to the fact that we do not know apriori whether the system of subspaces $E_I$ is complete. However it does not matter for the proof of uniform minimality. To be able to apply all definition we need to restrict ourself to the closed linear span of $E_I$, and the norms of spectral projections $\mathcal{E}^I$ can only decrease.

The following theorem is the main result of this chapter

**Theorem 8.1.** *Let $\varphi$, $\psi$ satisfy conditions 1 — 4 above, $W$ be a matrix $d \times d$ Muckenhoupt weight. Then the system of subspaces $E_I$ constructed above forms an unconditional basis in $L^2(W)$.*

The proof of the theorem consists of several steps. The following theorem is one of the main steps.

**Theorem 8.2.** *Let $W$ be a Muckenhoupt weight. Then the operator $\mathcal{J} : \ell^2(\mathbb{C}^d) \to L^2$,*

$$\mathcal{J}\{\boldsymbol{f}_I\} = \sum_{I \in \mathcal{D}} \varphi_I W^{1/2} (W_I)^{-1/2} \boldsymbol{f}_I$$

*is bounded (with norm estimated by a constant depending on $\varphi$, $d$ and the Muckenhoupt constant of $W$)*

**Proof.** Let us recall that in Section 5 we have proved that the system of Haar subspaces $E^h_I$ form a Riesz basis in $L^2(W)$ if $W$ is a Muckenhoupt weight. Since $W$ and $W^{-1}$ are Muckenhoupt weight with the same Muckenhoupt constant, the system $E^h_I$, $I \in \mathcal{D}$ is also a Riesz basis in $L^2(W^{-1})$, or, equivalently, the system $W^{-1/2}E^h_I$, $I \in \mathcal{D}$ is a Riesz basis in $L^2$

Any vector in the subspace $E_I[\chi]$ can be represented as $h_I\boldsymbol{e}$, where $h_I$ is the normalized Haar function, and $\boldsymbol{e} \in \mathbb{C}^d$. It is more convenient to represent an arbitrary function in $E_I[\chi]$ as $f = h_I \left([W^{-1}]_I\right)^{-1/2} \boldsymbol{e}$; in this case $\|f\|_{L^2(W)} = \|\boldsymbol{e}\|_{\mathbb{C}^d}$.

Let $\tilde{\boldsymbol{e}}^k_I$, $k = 1, 2, \ldots, d$ be an orthonormal basis of eigenvectors of $\left([W^{-1}]_I\right)^{-1/2}$. Then the system $h_I W^{-1/2} \left([W^{-1}]_I\right)^{-1/2} \tilde{\boldsymbol{e}}^k_I$, $k = 1, 2, \ldots, d$ is an orthonormal basis in $W^{-1/2}E_I[\chi]$. So, the system $\{g^k_I\}$

$$g^k_I = h_I W^{-1/2} \left([W^{-1}]_I\right)^{-1/2} \tilde{\boldsymbol{e}}^k_I, \qquad I \in \mathcal{D}\,,\ k = 1, 2, \ldots, d$$

is a Riesz basis in $L^2$.

To prove that the operator $\mathcal{J}$ is bounded it is sufficient to take an orthogonal basis $f_I^k$ in $\ell^2(\mathbb{C}^d)$ and show that the matrix of $\mathcal{J}$

$$(\mathcal{J} f_J^l, g_I^k),, \qquad I, J \in \mathcal{D}, \ k, l = 1, 2, \ldots, d$$

generates a bounded operator in $\ell^2$. Let vectors $f_I^k \in \ell^2(\mathbb{C}^d)$ be a natural (for our purposes) orthonormal basis in $\ell^2(\mathbb{C}^d)$, namely, $I$th component of $f_I^k$ is $\boldsymbol{e}_I^k$, and all other components are 0, then the matrix can be rewritten as

$$(\varphi_J, h_I)_{L^2}(W_J^{-1/2}\boldsymbol{e}_J^l, \left([W^{-1}]_I\right)^{-1/2}\tilde{\boldsymbol{e}}_I^k), \qquad I, J \in \mathcal{D}, \ k, l = 1, 2, \ldots, d.$$

Grouping $d \times d$ blocks we get

$$(\varphi_J, h_I)_{L^2}\left[(W^{-1})_I\right]^{-1/2}(W_J)^{-1/2}, \qquad I, J \in \mathcal{D},$$

and we have to prove that this matrix generates a bounded operator in $\ell^2(\mathbb{C}^d)$.

By Corollary 3.3 $(W_I)^{-1} \leq (W^{-1})_I$, and similarly $[(W^{-1})_I]^{-1} \leq W_I$, therefore

$$\left\| \left[(W^{-1})_I\right]^{-1/2}(W_J)^{-1/2} \right\| \leq \left\| (W_I)^{1/2}([W^{-1}]_J)^{1/2} \right\|.$$

We want to prove that the matrix

$$|(\varphi_J, h_I)_{L^2}| \left\| (W_I)^{1/2}([W^{-1}]_J)^{1/2} \right\|, \qquad I, J \in \mathcal{D} \tag{8.1}$$

generates a bounded operator in $\ell^2$ (of course it is sufficient).

The scalar product $|(\varphi_J, h_I)_{L^2}|$ can be estimated as

$$|(\varphi_J, h_I)_{L^2}| \leq C\frac{|I|^{3/2}}{|J|^{3/2}}\frac{|J|^2}{\left|\lambda(I) - \overline{\lambda(J)}\right|^2} \qquad \text{for } |I| \leq |J| \tag{8.2}$$

If $|I| \geq |J|$ one can write another estimate. Let as in Section 7 $Z(I)$ denotes the interval $3I$ (the interval $I$ itself an the two neighbors of the same size), $s(I)$ stands for the three point set formed of the center point and endpoints of $I$, and $\rho_{J,I}$ be the distance from $\lambda(J)$ to $s(I)$. Then for $|I| \geq |J|$

$$|(\varphi_J, h_I)_{L^2}| \leq C\frac{|J|^{3/2}}{|I|^{3/2}}\frac{|I|^2}{\left|\lambda(I) - \overline{\lambda(J)}\right|^2}, \qquad \text{if } J \not\subset Z(I) \tag{8.3}$$

and

$$|(\varphi_J, h_I)_{L^2}| \leq C\frac{|J|^{3/2}}{|I|^{1/2}} \cdot \frac{1}{\rho_{J,I}}, \qquad \text{if } J \subset Z(I) \tag{8.4}$$

Then the matrix (8.1) is bounded above by a sum of matrices from Lemmas 7.7, 7.8 (modulo changing $I$ to $J$ and $W$ to $W^{-1}$ in some places), so the corresponding operator is bounded.

To complete the proof of the theorem we need to explain the estimates (8.2) — (8.4). The first one is easiest to explain:

$$|(\varphi_J, h_I)_{L^2}| \le \sup_I |\varphi'(t)| \cdot |I|^{3/2},$$

and by property 2 of $\varphi$

$$\sup_I |\varphi'(t)| \le C|J|^{-3/2} \frac{|J|^2}{\left|\lambda(I) - \overline{\lambda(J)}\right|^2},$$

that gives us the required estimate.

The estimate (8.3) is also fairly simple: first of all

$$|(\varphi_J, h_I)_{L^2}| \le \sup_I |\varphi(t)| \cdot |I|^{1/2}.$$

Since $J \not\subset Z(I)$, we have

$$\sup_I |\varphi(t)| \le C \sup_I \frac{|J|^{3/2}}{\left|t - \overline{\lambda(J)}\right|^2} \asymp \frac{|J|^{3/2}}{\left|\lambda(I) - \overline{\lambda(J)}\right|^2},$$

and above two estimates imply (8.3).

To prove (8.4) let first estimate $\int_I \varphi_J$. Suppose $J \subset I = [a, b]$. Then, since the integral $\int_R \varphi_J = 0$ we can apply property 3

$$\left|\int_I \varphi_J\right| = \left|\int_{-\infty}^a \varphi_J + \int_b^\infty \varphi_J\right| \le C \frac{|J|^{3/2}}{\min\{|\lambda(J) - a|, |\lambda(J) - b|\}} \tag{8.5}$$

If $J \not\subset I$ (and $|J| \le |I|$) then $I$ is strictly on the left or on the right side of $J$. Suppose (without loss of generality) the later. Then we can write $\int_I \varphi_J = \int_a^\infty \varphi_J - \int_b^\infty \varphi_J$, and again the estimate (8.5) holds.

So

$$|(\varphi_J, h_I)_{L^2}| \le \frac{|J|^{3/2}}{\rho_{J,I}} \frac{1}{|I|^{1/2}}$$

and we are done. □

**Theorem 8.3.** *For an arbitrary numerical sequence $\alpha = \{\alpha_I\}_{I \in \mathcal{D}}$, the operator $\mathcal{E}_\alpha$,*

$$\mathcal{E}_\alpha f = \sum_{I \in \mathcal{D}} \alpha_I \mathcal{E}^I f$$

*is a bounded operator in $L^2(W)$ and its norm is bounded by $\|\{\alpha_I\}\|_\infty$.*

In particular this theorem implies that the system of subspaces $E_I$, $I \in \mathcal{D}$ forms a Riesz basis in the closure of its linear span in $L^2(W)$.

**Proof of Theorem 8.3.** The operator $\mathcal{E}_\alpha$ in $L^2(W)$ is unitarily equivalent to the operator $W^{1/2}\mathcal{E}_\alpha W^{-1/2}$ in $L^2$. If $\mathcal{J}_W^\varphi$ denotes the operator from Theorem 8.2 $\mathcal{J}_W^\varphi : \ell^2(\mathbb{C}^d) \to L^2$,

$$\mathcal{J}_W^\varphi\{\boldsymbol{f}_I\} = \sum_{I\in\mathcal{D}} \varphi_I W^{1/2}(W_I)^{-1/2}\boldsymbol{f}_I\,,$$

the operator $\mathcal{E}_\alpha$ can be factorized as

$$\mathcal{E}_\alpha = \mathcal{J}_W^\varphi M(\mathcal{J}_{W^{-1}}^\psi)^*\,,$$

where $M$ is the multiplication by the sequence $\alpha_I(W_I)^{1/2}([W^{-1}]_I)^{1/2}$ in $\ell^2(\mathbb{C}^d)$,

$$M\{\boldsymbol{f}_I\}_{I\in\mathcal{D}} = \left\{\alpha_I\cdot(W_I)^{1/2}([W^{-1}]_I)^{1/2}\boldsymbol{f}_I\right\}_{I\in\mathcal{D}}\,.$$

Because of Muckenhoupt condition the operator $M$ is bounded, and by Theorem 8.2 operators $\mathcal{J}_W^\varphi$ and $\mathcal{J}_{W^{-1}}^\psi$ are bounded too. □

The following theorem shows that the system of subspaces $E_I$, $I\in\mathcal{D}$ is a complete system in $L^2(W)$. Note, that it does not follows automatically from the fact that the system is complete in $L^2$ (completeness of polynomials in weighted $L^2$-spaces on the real line $\mathbb{R}$ is a nontrivial question, for example).

**Theorem 8.4.** *If $W$ is a $d\times d$ Muckenhoupt weight, the system of subspaces $E_I$, $I\in\mathcal{D}$ is complete in $L^2(W)$.*

**Proof.** Since the system $E_I$, $I\in\mathcal{D}$ is a Riesz basis in $L^2$, then for any $f\in L^2$ the series

$$\sum_{I\in\mathcal{D}} \mathcal{E}^I f$$

converges unconditionally (partial sums converge independently of order) in the norm of $L^2$ to $f$.

Theorem 8.3 implies that for any $f\in L^2(W)$ partial sums of

$$\sum_{I\in\mathcal{D}} \mathcal{E}^I f \tag{8.6}$$

are uniformly bounded. Since $E_I$, $I\in\mathcal{D}$ is a Riesz basis (an isomorphic image of an orthogonal basis) in the closure of its linear span in $L^2(W)$, implies that the series (8.6) converges (in $L^2(W)$) for any $f\in L^2(W)$.

Let $\mathcal{H} = L^2\cap L^2(W)$. Clearly $\mathcal{H}$ is dense in both $L^2$ and $L^2(W)$. Then for any $f\in\mathcal{H}$ the series (8.6) converges in norms of both $L^2$ and $L^2(W)$. The imbedding operator from $L^2$ to $L^2(W)$ (with natural domain) is a closed operator — this is a well-known fact that follows, for example, from the fact that for any sequence that converges in $L^2$ one can find a subsequence that converges almost everywhere. Therefore the series (8.6) converges to $f$ in both norms, so $\mathcal{H}$ is in the $L^2(W)$ closed linear span of $E_I$, $I\in\mathcal{D}$. □

**Remark.** The results of this section can be extended to the more general case of wavelet transform, when we do not assume that the system $\varphi_I$, $I\in\mathcal{D}$ is a Riesz basis or even uniformly minimal. We only assume that the functions $\varphi$, $\psi$ satisfy conditions 1 — 4 above, and that the series

$$\sum_{I\in\mathcal{D}} \varphi_I (f, \psi_I)_{L^2} \stackrel{\text{def}}{=} \sum_{I\in\mathcal{D}} \mathcal{E}^I f$$

converges to $f$ for all $f$ in a dense set in $L^2$. Theorems 8.2, 8.3, 8.4 hold and all proofs work in this case. Of course in Theorem 8.3 $\mathcal{E}^I$ denote not spectral projections but simply operators given by the formula

$$\mathcal{E}^I f = \varphi_I (f, \psi_I)_{L^2} = \varphi_I \cdot \int_{\mathbb{R}} f\overline{\psi_I}\,.$$

Theorem 8.1 can be easily reformulated as follows: *If $W$ is a $d\times d$ matrix Muckenhoupt weight, then for any $f \in L^2(W)$ the series*

$$\sum_{I\in\mathcal{D}} \varphi_I (f, \psi_I)_{L^2} \stackrel{\text{def}}{=} \sum_{I\in\mathcal{D}} \mathcal{E}^I f$$

*converges unconditionally (independently of order) to $f$ in the norm on $L^2(W)$.*

All the proofs remain practically the same.

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