LINEAR RESOLVENT GROWTH OF A WEAK CONTRACTION DOES NOT IMPLY ITS SIMILARITY TO A NORMAL OPERATOR.

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Abstract. It was shown in [2] that if T is a contraction in a Hilbert space with finite defect ($\|T\| \le 1$, $\operatorname{rank}(I-T^*T) < \infty$), and its spectrum $\sigma(T)$ doesn't coincide with the closed unit disk $\overline{\mathbb{D}}$, then the following Linear Resolvent Growth condition

$$\|(\lambda I - T)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda, \sigma(T))}, \ \lambda \in \mathbb{C} \backslash \sigma(T),$$

implies that T is similar to a normal operator.

The condition $\operatorname{rank}(I - T^*T) < \infty$ characterizes how close is T to a unitary operator. A natural question arises about relaxing this condition. For example, it was conjectured in [2] that one can replace the condition $\operatorname{rank}(I - T^*T) < \infty$ by $I - T^*T \in \mathfrak{S}_1$, where $\mathfrak{S}_1$ denotes the trace class.

In this note we show that this conjecture is not true, moreover it is impossible to replace the condition $\operatorname{rank}(I - T^*T) < \infty$ by any reasonable condition of closedness to a unitary operator. For example, we construct a contraction T (i. e. $\|T\| \le 1$), $\sigma(T) \ne \overline{\mathbb{D}}$, satisfying $I - T$, $I - T^*T$, $I - TT^* \in \mathfrak{S} := \cap_{p>0} \mathfrak{S}_p$, where $\mathfrak{S}_p$ stands for the Schatten–von-Neumann class, satisfying the above Linear Resolvent Growth condition but not similar to a normal operator.

Notation

$\mathbb{D}$	the unit disk $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$ in the complex plane $\mathbb{C}$;
$s_n(A)$	singular number of the operator A, $$s_n(A) = \inf\{\|A - K\| \, : \, \operatorname{rank} K \le n\},$$ $s_0(A) = \|A\|$. For a compact operator A, the sequence $s_k(A)^2$, $k = 0, 1, 2, \dots$ is exactly the system of eigenvalues of A^*A (counting multiplicities) taken in decreasing order;
$\mathfrak{S}_p$	the Schatten–von-Neumann class of compact operators A such that $\sum_{k=1}^{\infty} s_k(A)^p < \infty, p > 0$.
$\|A\|_{\mathfrak{S}_p}$	the norm in $\mathfrak{S}_p$, $\|A\|_{\mathfrak{S}_p} := \left(\sum_0^{\infty} s_n(A)^p\right)^{1/p}$

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0. Introduction and main results

In this note we are dealing with the question of similarity to a normal operator. Let us recall that operators A and B are similar if there exists a (bounded) invertible operator R such that $A = RBR^{-1}$. Similarity of an operator T to a normal operator means that the operator T admits a reach functional calculus, for example that $f(T)$ is well defined for any continuous function f on the complex plane $\mathbb{C}$.

We continue with a small deviation in the history of the question. Probably, the article [11] was the first work where a criterion on the similarity of a contraction to a unitary operator was given (we recall that an operator T is called a contraction if $\|T\| \le 1$). Then the result of the work was transformed in a resolvent test by I. Gohberg and M. Krein, [6]. The next advance in the subject was due to work by N. Nikolski and S.V. Khruschev [9]. The authors managed to get the counterpart of the previous result for contractions with spectra inside the unit disk $\mathbb{D}$ and defect operators of rank one. In the work [2] the generality of the test was pushed up to contractions of arbitrary finite defects.

Since for a normal operator N the norm of the resolvent can be computed as

$$\|(N-\lambda I)^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(N))},$$

the condition

$$\|(T-\lambda I)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda, \sigma(N))} \tag{0.1}$$

is necessary for the operator T to be similar to a normal operator.

The condition (0.1) above, which we will call *Linear Resolvent Growth* condition (LRG), is clearly not sufficient for the similarity to a normal operator: multiplication by the independent variable z on the Hardy space H^2 clearly satisfies (0.1) but there is no similarity we search for.

However, if the spectrum of an operator is "thin", and the operator is close to a "good" operator, one can expect that the Linear Resolvent Growth (0.1) is sufficient for the similarity to a normal operator.

It was proved in the work [2], that if a contraction T is close to a unitary operator in the sense that it has a finite rank defect $I - T^*T$, and its spectrum doesn't coincide with the closed unit disk $\overline{\mathbb{D}}$, then the Linear Resolvent Growth (0.1) implies similarity to a normal operator.

It was also shown there that for a contraction T the condition $I - T^*T \in \mathfrak{S}_p$, $p > 1$, where $\mathfrak{S}_p$ stands for the Schatten–von-Neumann class, is not sufficient, and it was conjectured, that the condition $I - T^*T \in \mathfrak{S}_1$ (plus the assumption that the spectrum is not the whole closed unit disk $\overline{\mathbb{D}}$) guarantees the equivalence of LRG and similarity to a normal operator.

We show in this note, that this is not the case, i.e., one can find a contraction T, with simple countable spectrum, satisfying $I - T^*T \in \mathfrak{S}_1$ (or even

$I - T^*T \in \cap_{p>0}\mathfrak{S}_p$) and satisfying the Linear Resolvent Growth condition (0.1), but not similar to a normal operator.

Moreover, we will show, that no reasonable condition of closedness to a unitary operator (except of finite rank defect $I - T^*T$) implies that the Linear Resolvent Growth is equivalent to the similarity to a normal operator.

Let us explain what do we mean by a "reasonable" condition. Suppose we have a function Φ with values in $\mathbb{R}_+ \cup \{\infty\}$ (which measures, how small an operator (defect) is) defined on a non-negative operators in a Hilbert space H, satisfying $\Phi(\mathbf{0}) = 0$ and such that

1. Φ is increasing, i. e. $\Phi(A) \le \Phi(B)$ if $A \le B$;
2. $\Phi(A) < \infty$ if $\operatorname{rank} A < \infty$;
3. Φ is upper semicontinuous, i. e. if $A_n \nearrow A \quad (A_n \le A, ||A_n - A|| \to 0)$ then $\Phi(A) \le \lim_n \Phi(A_n)$;
4. Φ is lower semicontinuous in the following weak sense: if $\operatorname{rank} A < \infty$ and $\operatorname{rank} A_n \le N$ (for some $N < \infty$), $\lim_n \|A_n\| = 0$, then $\lim_n \Phi(A \oplus A_n) = \Phi(A)$ (here $A \oplus B$ means that $\operatorname{range} A \perp \operatorname{range} B$ and $(\operatorname{Ker} A)^\perp \perp (\operatorname{Ker} B)^\perp$).

We extend the function Φ to non-selfadjoint operators by putting $\Phi(A) := \Phi((A^*A)^{1/2})$.

We have in mind the following examples of functions Φ:

1. $\Phi(A) = \|A\|_{\mathfrak{S}_p} = \left(\sum s_n(A)^p\right)^{1/p}$, where $s_n(A)$ is nth singular value of the operator A. In this case $\Phi(A) < \infty$ means exactly $A \in \mathfrak{S}_p$
2. $\Phi(A) := \sum_{n=1}^\infty 2^{-n} \frac{\|A\|_{\mathfrak{S}_{1/n}}}{1+\|A\|_{\mathfrak{S}_{1/n}}}$; in this case $\Phi(A) < \infty$ if and only if $A \in \bigcap_{p>0} \mathfrak{S}_p$;
3. Any weighted sum of singular numbers, for example
$$\Phi(A) = \sum_1^\infty 2^{2^n} s_n(A).$$
4. The function Φ_ψ,
$$\Phi_\psi(A) := \sum_0^\infty \psi(s_n(A)),$$
where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasung and continuous at 0 function, satisfying $\psi(0) = 0$.

   The condition $\Phi_\psi(A) < \infty$ characterizes the class $\mathfrak{S}_\psi$, introduced in [2], $A \in \mathfrak{S}_\psi \iff \Phi_\psi(A) < \infty$. Note that if we allow $\psi(0)$ to be positive, then for any ψ satisfying $\psi(0) > 0$ the class $\mathfrak{S}_\psi$ is just the ideal of finite rank operators.

Our main result is the following theorem.

Theorem 0.1. *Let Φ be a function satisfying the conditions 1–4 above. Given $\varepsilon > 0$, there exists a contraction T (i. e. $\|T\| \le 1$) on a Hilbert space H such that:*

1. *The spectrum $\sigma(T)$ is a countable subset of the closed unit disk $\overline{\mathbb{D}}$;*
2. $T = I + K$*, where* $\Phi(K)$, $\Phi(K^*) \le \varepsilon$*;*
3. $\Phi(I - T^*T) \le \varepsilon$, $\Phi(I - TT^*) \le \varepsilon$*;*
4. T *satisfies the Linear Resolvent Growth condition*
$$\|(T - \lambda I)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda, \sigma(T))} ,$$
but
5. T *is not similar to a normal operator*

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1. Proof of the main result

1.1. **Preliminaries about bases.** Before going to the proof, we need to remind the reader some well-known facts about bases in a Hilbert space.

The results discussed in this subsection are of common knowledge. The exhausting information on the subject may be found in the monograph [8], pages 131–133, 135–142. We should mention in this connection works [13, 14, 15] also.

Let $\{f_n\}_1^\infty$ be a complete system of vectors in a Hilbert space H. The system is called a *basis* if any vector $f \in H$ admits a unique decomposition
$$f = \sum_1^\infty c_n f_n,$$
where the series converges (in the norm of H), and the system is called an *unconditional basis* if it is a basis and the series converges unconditionally (for any reordering).

A complete system is called a Riesz basis if it is equivalent to the orthonormal basis, i. e. if there exists a bounded invertible operator R (the so-called *orthogonalizer*) such that $Rf_n = e_n \ \forall n$, where $\{e_n : n = 1, 2, \dots\}$ is some orthonormal basis. Clearly, an orthogonalizer is unique up to unitary factor on the left.

The quantity $r(\{f_n\}) := \|R\| \cdot \|R^{-1}\|$ is therefore well defined, and could serve as the measure of non-orthonormality of the Riesz basis $\{f_n\}$.

Clearly, a Riesz basis is an unconditional basis. Notice, (although we do not need that in what follows), that the converse is also true: an theorem due to Köthe and Toeplitz states that a normalized ($0 < \inf \|f_n\| \le \sup \|f_n\| < \infty$) unconditional basis is a Riesz basis.

Let us also mention the connection between Riesz bases and similarity to normal operators. It is just a trivial observation, that if T is an operator with simple eigenvalues and with a complete system of eigenvectors f_n, $n = 1, 2, \dots$, then T is similar to a normal operator if and only if the system of eigenvectors is a Riesz basis. In this case the similarity transformation is given by an orthogonalizer R, namely RTR^{-1} is a normal operator.

1.2. **Global construction.** Suppose we have constructed a sequence of finite rank operators $A_n : \mathbb{C}^n \to \mathbb{C}^n$, with simple spectrum and let $\{f_k^n\}_{k=1}^n$ be the system of (normalized, $\|f_k^n\| = 1$) eigenvectors of A_n. Suppose the operators A_n (note that we do not assume even that A_n are contractions here) satisfy the following properties:

1. Operators A_n satisfy uniformly the Linear Resolvent Growth condition, i. e.
$$\|(A_n - \lambda I)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda, \sigma(A_n))} ,$$
where the constant C doesn't depend on n;
2. $\lim_n r(\mathcal{F}_n) = \infty$, where $r(\mathcal{F}_n) = \|R_{\mathcal{F}_n}\| \cdot \|R_{\mathcal{F}_n}^{-1}\|$ is the measure of non-orthogonality of the system $\mathcal{F}_n = \{f_k^n\}_{k=1}^{N_n}$ of the eigenvectors of A_n (recall that $R_{\mathcal{F}_n}$ is the *orthogonalizer* of the system $\mathcal{F}_n$).

Then we are done with the proof of Theorem 0.1.

Let us explain why. We construct an operator $T = \oplus_{n=1}^\infty \big(a_n A_n + b_n I\big)$, where $|b_n| < 1$, $\lim_n b_n = 1$, $\lim_n a_n = 0$. We pick the numbers a_n, b_n in such a way, that the spectra of the summands $a_n A_n + b_n I$ do not intersect, so the resulting operator has the simple spectrum.

Since the linear transformation $A \mapsto aA + bI$ doesn't change the Linear Resolvent Growth condition, and, moreover, doesn't change the constant (we leave the proof of this fact as a simple exercise for the reader), the resulting operator T satisfies $\|(T - \lambda I)^{-1}\| \le C/\operatorname{dist}(\lambda, \sigma(T))$.

Since the same linear transformation doesn't change the system of eigenvectors, we can conclude that the system $\mathcal{F}$ of eigenvectors of T is the direct sum of eigenvectors of all A_n, i. e. $\mathcal{F} := \oplus_{n=1}^\infty \mathcal{F}_n$.

Since $r(\mathcal{F}_n) \to \infty$ (see property 2 of A_n), the system $\mathcal{F}$ of eigenvectors of T is not a Riesz basis, and therefore (since T has simple spectrum), T is not similar to a normal operator.

It remains to show, that one can chose numbers a_n, b_n such, that the operator T is close to a unitary operator, namely $\Phi(I - T) \le \varepsilon$, $\Phi(I - T)^* \le \varepsilon$, $\Phi(I - T^*T) \le \varepsilon$, $\Phi(I - TT^*) \le \varepsilon$.

We will construct the numbers a_n, b_n by induction. Note, that the inequality $|a_n| \cdot \|A_n\| \le 1 - |b_n|$ implies that the operator $T_n = a_n A + b_n I$ is a strict contraction ($\|T_n\| < 1$).

So, we will always take a_n satisfying $|a_n| \cdot \|A_n\| < 1 - |b_n|$. Under this assumption
$$\|I - T_n\| < 1 - |b_n| + |1 - b_n| \le 2 \cdot |1 - b_n|.$$
The simple identity $(I - \Delta)^*(I - \Delta) = I - \Delta - \Delta^* - \Delta^*\Delta$ (applied to $\Delta = I - T_n$, $\Delta = I - T_n^*$) implies that in this case
$$\|I - T^*T\|, \|I - TT^*\| < 6 \cdot |1 - b_n|,$$
if $|1 - b_n| \le 1/2$.

Therefore, picking b_n close to 1 (and a_n satisfying $|a_n| \cdot \|A_n\| < 1 - |b_n|$) we can make the norms of the finite rank operators $I - T_n$, $I - T_n^* T_n$, $I - T_n T_n^*$, where $T_n = a_n A_n + b_n I$, as small as we want.

Since $\Phi(\mathbf{0}) = 0$, property 4 of Φ implies that we can pick a contraction $T_1 = a_1 A_1 + b_1 I$ such that

$$\Phi(I - T_1) \le \varepsilon/2, \qquad \Phi(I - T_1)^* \le \varepsilon/2$$
$$\Phi(I - T_1^* T_1) \le \varepsilon/2, \qquad \Phi(I - T_1 T_1^*) \le \varepsilon/2.$$

Suppose we constructed the finite rank contractions $T_k = a_k A_k + b_k I$, $k = 1, 2, \dots, n-1$ such that the operator $T^{(n-1)} = T_1 \oplus T_2 \oplus \ldots \oplus T_{n-1}$, $\|T^{(n-1)}\| < 1$, has simple spectrum and satisfies

$$\begin{aligned} \Phi(I - T^{(n-1)}) &\le (1 - 2^{-(n-1)})\varepsilon, \\ \Phi(I - T^{(n-1)*}) &\le (1 - 2^{-(n-1)})\varepsilon, \\ \Phi(I - T^{(n-1)*} T^{(n-1)}) &\le (1 - 2^{-(n-1)})\varepsilon, \\ \Phi(I - T^{(n-1)} T^{(n-1)*}) &\le (1 - 2^{-(n-1)})\varepsilon. \end{aligned}$$

Making the norm $\|I - T_n\|$ sufficiently small we can guarantee that the operator $T^{(n)} = T_1 \oplus T_2 \oplus \ldots \oplus T_n$ has simple spectrum, satisfies $\|T^{(n)}\| < 1$ and property 4 of Φ implies that one can make $T^{(n)}$ to satisfy

$$\begin{aligned} \Phi(I - T^{(n)}) &\le (1 - 2^{-n})\varepsilon, \\ \Phi(I - T^{(n)*}) &\le (1 - 2^{-n})\varepsilon, \\ \Phi(I - T^{(n)*} T^{(n)}) &\le (1 - 2^{-n})\varepsilon, \\ \Phi(I - T^{(n)} T^{(n)*}) &\le (1 - 2^{-n})\varepsilon. \end{aligned}$$

Property 3 of Φ implies that the operator $T = \oplus_{n=1}^{\infty} T_n$ satisfies

$$\Phi(I - T) \le \varepsilon, \qquad \Phi(I - T^*) \le \varepsilon$$
$$\Phi(I - T^* T) \le \varepsilon, \qquad \Phi(I - T T^*) \le \varepsilon.$$

We are done (modulo the constructing of A_n) $\square$

1.3. **More preliminaries about bases.** We will need some more information about bases. Let f_n, $n = 1, 2, \dots$ be a linearly independent sequence of vectors. Let P_n denote the projection onto first n vectors of the system, $P_n \sum c_k f_k = \sum_1^n c_k f_k$ (the operators P_n are well defined on finite linear combinations of f_k). The following characterization of bases is well-known, see for example the monographs [12], page 46–47 or [16], pages 37–39.

Theorem 1.1 (Banach Basis Theorem)**.** *A complete system of vectors f_k, $k = 1, 2, \dots$ is a basis if and only if* $\sup_n \|P_n\| =: K < \infty$.

If one *a priori* assumes that the projections P_n are bounded, then the theorem is just the Banach–Steinhaus Theorem.

We will need the following corollary characterizing the bases in terms of so-called *multipliers*.

For a numerical sequence $\alpha := \{\alpha_n\}_1^\infty$, let M_α be a *multiplier*, defined by

$$M_\alpha f_n = \alpha_n f_n, \qquad n = 1, 2, \dots$$

(a priory M_α is defined only on finite linear combinations $\sum c_k f_k$).

For a sequence α its variation $\operatorname{var}(\alpha)$ is defined as

$$\operatorname{var}\alpha := \sum_1^\infty |a_k - a_{k+1}|.$$

Clearly, if $\operatorname{var}\alpha < \infty$, the limit $\lim_n \alpha_n =: \alpha_\infty$ exists and is finite.

Corollary 1.2. *Let a system of vectors f_n, $n = 1, 2, \dots$ be a basis. If for a numerical sequence $\alpha = \{\alpha_n\}_1^\infty$ we have $\operatorname{var}\alpha < \infty$, then*

$$\|M_\alpha\| \le K \operatorname{var}\alpha + |\alpha_\infty|,$$

where K is the constant from the Banach Basis Theorem (Theorem 1.1), and $\alpha_\infty := \lim_n \alpha_n$.

Proof. The proof follows immediately from the formula

$$M_\alpha = \sum_{n=1}^\infty (\alpha_n - \alpha_{n+1}) P_n + \alpha_\infty I,$$

where P_n are the projections from the Banach Basis Theorem. □

Remark 1.3. The above Corollary 1.2 holds for bases in finite-dimensional spaces as well: one just have to extend the finite sequence α by zeroes.

Remark 1.4. We do not need this fact, but let us just mention, that the converse of Corollary 1.2 is also true. Namely, a system of vectors f_n, $n = 1, 2, \dots$ is a basis if and only if for any numerical sequence α of bounded variation,the corresponding multiplier M_α is bounded. The proof is quite easy, cf. [8, 12].

1.4. **Construction of the operators A_n.** To construct the operators A_n from Section 1.2, consider a normalized ($\|f_n\| = 1$) system of vectors $\mathcal{F} := \{f_n\}_1^\infty$ which is basis but not a Riesz basis. Such systems do exist, see an example below in Section 2.

For the system $\mathcal{F}$ its measure of non-orthogonality

$$r(\mathcal{F}) := \|R_\mathcal{F}\| \cdot \|R_\mathcal{F}^{-1}\| = \infty.$$

Therefore, for finite truncations $\mathcal{F}_n = \{f_k\}_{k=1}^n$ we have

$$r(\mathcal{F}_n) := \|R_{\mathcal{F}_n}\| \cdot \|R_{\mathcal{F}_n}^{-1}\| \to \infty \qquad \text{as } n \to \infty.$$

We define operators A_n as follows. Let $\{\lambda_n\}_1^\infty$ be a strictly increasing sequence of real numbers. Define an operator A_n on $\mathcal{L}\{f_k : k = 1, \dots N_n\}$ by $A_n f_k = \lambda_k f_k$. It is easy to see that the operator A_n has simple spectrum, and that Property 2 of A_n is satisfied.

We have to show that Property 1 holds, i. e. that

$$\|(A_n - \lambda I)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda, \sigma(A_n))} .$$

To estimate the norm $\|(A_n - \lambda I)^{-1}\|$ we will use Corollary 1.2. Namely, if we put $\alpha := \{\alpha_k\}_1^\infty$, where

$$\alpha_k = \begin{cases} (\lambda_k - \lambda)^{-1}, & k \le n \\ 0 & k > n\ , \end{cases}$$

then

$$\|(A_n - \lambda I)^{-1}\| \le \|M_\alpha\| \le K \cdot \operatorname{var} \alpha.$$

So, we need to show that

$$\operatorname{var} \alpha \le \frac{C}{\operatorname{dist}(\lambda, \sigma(A_n))}\ .$$

Let $\lambda_m \le \operatorname{Re}\lambda < \lambda_{m+1}$ (we consider the trivial cases $\operatorname{Re}\lambda < \lambda_1$ or $\operatorname{Re}\lambda \ge \lambda_n$ later). Then

$$\operatorname{var}\alpha = \sum_{k=1}^{m-1} |\alpha_k - \alpha_{k+1}| + \sum_{k=m+1}^{n-1} |\alpha_k - \alpha_{k+1}| \quad + \quad |\alpha_m - \alpha_{m+1}| + |\alpha_n|.$$

The last two terms are easy to estimate:

$$|\alpha_m - \alpha_{m+1}| + |\alpha_n| \le |\alpha_m| + |\alpha_{m+1}| + |\alpha_n| \le \frac{3}{\operatorname{dist}(\lambda, \sigma(A_n))}\ .$$

Let us estimate the first sum:

$$\begin{aligned} \sum_{k=1}^{m-1} |\alpha_k - \alpha_{k+1}| &\le \sum_{k=1}^{m-1} \left| \frac{1}{\lambda_k - \lambda} - \frac{1}{\lambda_{k+1} - \lambda} \right| \\ &= \sum_{k=1}^{m-1} \left| \int_{\lambda_k}^{\lambda_{k+1}} \frac{dz}{(z-\lambda)^2} \right| \le \int_{\lambda_1}^{\lambda_m} \frac{dz}{|z-\lambda|^2} \le \frac{C}{|\lambda - \lambda_m|}\ . \end{aligned}$$

Similarly,

$$\sum_{k=m+1}^{n-1} |\alpha_k - \alpha_{k+1}| \le \frac{C}{|\lambda - \lambda_m|}\ ,$$

and we are done. Note, that in trivial cases ($\operatorname{Re}\lambda < \lambda_1$ or $\operatorname{Re}\lambda \ge \lambda_n$) we just need to estimate one sum using integrals. □

Remark 1.5. The fact that the operators A_n satisfy the Linear Resolvent Growth condition follows immediately from a more general result about operators with spectrum on Ahlfors curves, proved in [2]. We presented the proof only for the reader's convenience.

Notice also, that everything above would work if we consider different monotone sequences $\{\lambda_k^n\}_{k=1}^n$, $n = 1, 2, \dots$, and put $A_n f_n := \lambda_k^n f_n$.

2. Nontrivial conditional basis

Let us consider space $L^2(w)$, $w(t)$ be a nonnegative measurable function on the unit circle $\mathbb{T} = \partial\mathbb{D}$,

$$\|f\|^2_{L^2(w)} := \int_{-\pi}^{\pi} |f(e^{it})|^2 w(e^{it}) \frac{dt}{2\pi}\,.$$

We study properties of the system of exponents $\{z^n\}_{n=0}^{\infty}$. The following facts hold true.

Proposition 2.1 (see [15])**.** *The system of exponents $\{z^n\}_{n=0}^{\infty}$ in its own closed linear span in $L^2(w)$ is*

1. *Basis if and only if the weight w satisfies Muckenhoupt (A_2) condition,*

$$\sup_I \left(\frac{1}{|I|}\int_I w\right)\cdot\left(\frac{1}{|I|}\int_I w^{-1}\right) < \infty;$$

2. *Unconditional (Riesz) basis iff $w \in L^\infty(\mathbb{T}), 1/w \in L^\infty(\mathbb{T})$.*

Direct computations show that a weight with power singularity, say $w(z) = |z-1|^\alpha$ satisfies the Muckenhoupt (A_2)-condition if and only if $-1 < \alpha < 1$. By picking any non-zero α in this interval we get an example of a basis which is not an unconditional (Riesz) basis.

Proof of Proposition 2.1. The statement is probably well-known, and we present the proof only for the reader's convenience.

By the Banach Basis Theorem (Theorem 1.1 above) the system $\{z^n\}_{n=0}^{\infty}$ is a basis if and only if the projections P_n, $P_n(\sum c_k z^k) = \sum_{k=0}^{n} c_k z^k$ are uniformly bounded.

Consider the so-called Riesz projection P_+, $P_+(\sum c_k z^k) = \sum_{k=0}^{\infty} c_k z^k$. Since for $f \in \mathcal{L}(z^n : n \geq 0)$

$$P_n f = f - z^{n+1} P_+(\overline{z}^{n+1} f),$$

and multiplication by the independent variable z is a unitary operator on $L^2(w)$, it is easy to show that the operators P_n are uniformly bounded (on the closed linear span of $\{z^n\}_{n=0}^{\infty}$ in $L^2(w)$) if and only if the operator P_+ is bounded on $L^2(w)$. The last is equivalent to the boundedness of the Hilbert Transform T, $T := -iP_+ i(I - P_+)$, and it is well known, see [7], [4], page 254 that T is bounded on $L^2(w)$ if and only if the weight w satisfies the Muckenhoupt (A_2)-condition.

To prove statement 2, let us notice, that the system of exponents is a Riesz basis if for any analytic polynomial $f = \sum_{k=0}^{N} c_k z^k$

$$c\|f\|^2_{L^2(w)} \leq \sum |c_k|^2 = \|f\|^2_{L^2} \leq C\|f\|^2_{L^2(w)}\,.$$

Since the multiplication by z is a unitary operator on $L^2(w)$, the last estimate should hold for for any *trigonometric* polynomial $f = \sum_{-N}^{N} c_k z^k$, which is possible if and only if w, $1/w \in L^\infty$. $\square$

3. Linear fractional transformations and the Linear Resolvent Growth condition

It is probably not completely clear from the construction, but the main reason for the result is that both Linear Resolvent Growth condition and similarity to a normal operator are "Möbius invariant", but the conditions like $I - T^*T \in \mathfrak{S}_p$ are not, if one pays attention to constants.

Let us clarify this statement a bit. First of all, let us notice, that if $T = RNR^{-1}$, then $\varphi(T) = R\varphi(N)R^{-1}$ for any analytic in a neighborhood of $\sigma(T)$ function φ. So, the similarity to a normal operator is preserved for $\varphi(T)$.

It turns out that the Linear Resolvent Growth is preserved for linear fractional transformations $\varphi(T) = (aT + bI)(cT + dI)^{-1}$.

Lemma 3.1. *Let $\varphi(z) = (az+b)/(cz+d)$ be a linear fractional transformation.*[1] *If an operator T (note that we do not require it to be a contraction) satisfies the Linear Resolvent Growth condition*

$$\|(T - \lambda I)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda, \sigma(T))} \ , \tag{3.1}$$

then

$$\|\varphi(T)\| \le 10C \sup_{z\in\sigma(T)} |\varphi(z)|.$$

Corollary 3.2. *Let $\varphi(z) = (az + b)/(cz + d)$ be a linear fractional transformation. If an operator T satisfies the Linear Resolvent Growth condition* (3.1)*, then the operator $\varphi(T)$ satisfies the same condition with constant $10C$, i.e.*

$$\|(\varphi(T) - \lambda I)^{-1}\| \le \frac{10C}{\operatorname{dist}\{\lambda, \sigma(\varphi(T))\}}.$$

Proof. Consider a function $\tau(z) := 1/(z - \lambda)$. The composition $\varphi_1 := \tau \circ \varphi$ is a linear fractional transformation (for example, because it is a conformal automorphism of the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \infty$). Therefore, Lemma 3.1 implies

$$\begin{aligned} \|(\varphi(T) - \lambda I)^{-1}\| &= \|\tau(\varphi(T))\| = \|\varphi_1(T)\| \\ &\le 10C \sup_{z\in\sigma(T)} |\tau(\varphi(z))| \\ &= 10C \sup_{w\in\varphi(\sigma(T))} |\tau(w)| = \frac{10C}{\operatorname{dist}\{\lambda, \varphi(\sigma(T))\}}. \end{aligned}$$

To complete the proof it remains to recall that according to the Spectral Mapping Theorem (see [3, Theorem VII.3.11]) $\sigma(\varphi(T)) = \varphi(\sigma(T))$ for any function φ analytic in a neighborhood of $\sigma(T)$. □

[1] we include degenerate cases $a = 0$, $c = 0$ as well

Proof of Lemma 3.1. The first trivial observation is that a *linear* transformation $T \mapsto aT + b$ preserves the Linear Resolvent Growth condition, and moreover, it preserves the constant. This is indeed trivial for the shift $T \mapsto T + bI$, and for the transformation $T \mapsto aT$ it follows from the following chain of estimates:

$$\|(aT - \lambda I)^{-1}\| = |a|^{-1} \left\| \left(T - \tfrac{\lambda}{a} I\right)^{-1} \right\| \le \frac{1}{|a|} \cdot \frac{C}{\operatorname{dist}(\frac{\lambda}{a}, \sigma(T))} = \frac{C}{\operatorname{dist}(\lambda, \sigma(aT))}.$$

Now let us prove the lemma. First of all consider the case when φ is a linear function. Since the LRG condition is preserved under linear transformations, we can assume without loss of generality that $\varphi(z) = z$. By Riesz–Dunford formula we have

$$T = \frac{1}{2\pi i} \int_\gamma z \cdot (zI - T)^{-1} dz$$

where γ is a contour surrounding $\sigma(T)$ in positive direction.

Take for γ the circle with center at 0 of radius R, $R > \rho(T)$, where $\rho(T) = \sup_{z \in \sigma(T)} |z|$ is the spectral radius of T. Then

$$\|T\| \le \frac{1}{2\pi} \cdot 2\pi R \cdot \rho(T) \cdot \frac{C}{R - \rho(T)} = \rho(T) \cdot \frac{CR}{R - \rho(T)} \ .$$

Taking the limit as $R \to \infty$ we get

$$\|T\| \le C\rho(T) = C \sup_{z \in \sigma(T)} |z|$$

Now let us consider another degenerate case, when φ is a proper rational function, i. e. $\varphi = a/(bz + c)$. In this case the conclusion of the lemma is just the Linear Resolvent Growth condition, so the conclusion trivially holds with the same constant C.

Finally let us consider the case of general position,

$$\varphi = \frac{az + b}{cz + d}, \qquad a \ne 0, \quad c \ne 0.$$

Let τ be a linear transformation of $\mathbb{C}$ which maps $-1 \mapsto -b/a$, $0 \mapsto -d/c$. Then $\varphi \circ \tau = \alpha \cdot (z - 1)/z$, $\alpha \in \mathbb{C}$.

So, since linear transformations preserve the Linear Resolvent Growth property, it is enough to deal with the case $\varphi = (z - 1)/z$. Let

$$\delta := \sup_{z \in \sigma(T)} |\varphi(z)| = \sup_{z \in \sigma(T)} \left| \frac{z - 1}{z} \right| \ .$$

Consider the case $\delta \ge 1/2$ first. One can write

$$\varphi(T) = \frac{1}{2\pi i} \int_\Gamma \varphi(z)(zI - T)^{-1} dz,$$

where $\Gamma = \gamma_r \cup \gamma_R$, where γ_r, γ_R are circles $|z| = r$ and $|z| = R$ in negative and positive directions respectively. We assume that $r \to 0$ and $R \to \infty$.

One can estimate

$$\left\|\int_{\gamma_R}\dots\right\| \le \lim_{R\to\infty}\frac{1}{2\pi}\cdot 2\pi R\cdot\frac{C}{R} = C$$

On the other hand

$$\left\|\int_{\gamma_r}\dots\right\| \le \lim_{r\to 0}\frac{1}{2\pi}\cdot 2\pi r\cdot\frac{1}{r}\cdot\frac{C}{\operatorname{dist}(0,\sigma(T))} = \frac{C}{\operatorname{dist}(0,\sigma(T))}\ .$$

One can easily see (the level sets of $|\varphi|$ can be computed explicitly) that the set $\{z : |\varphi(z)| \le \delta\}$ is contained outside the disk $\{z : |z| = 1/(1+\delta)\}$, so $\operatorname{dist}(0,\sigma(T)) \ge 1/(1+\delta)$. Therefore

$$\left\|\int_{\gamma_r}\dots\right\| \le C\cdot(1+\delta),$$

and so

$$\|\varphi(T)\| \le C\cdot(2+\delta) \le 5C\delta = 5C\sup_{z\in\sigma(T)}|\varphi(z)|,$$

if $\delta \ge 1/2$.

Now let us consider the case $\delta \le 1/2$. It is easy to compute that for $\delta < 1$ the level set $\{z : |\varphi(z)| \le \delta\}$ is the closed unit disk centered at $c = 1/(1-\delta^2)$ and of radius $r = \delta/(1-\delta^2)$. By the definition of δ, the spectrum $\sigma(T)$ is contained in this level set.

Let us write

$$\varphi(T) = \frac{1}{2\pi i}\int_\Gamma \varphi(z)(zI-T)^{-1}dz,$$

where Γ now is the circle centered at $c = 1/(1-\delta^2)$ of radius $\frac{3}{2}r$. We can estimate

$$\|\varphi(T)\| \le \lim_{r\to 0}\frac{1}{2\pi}\cdot 2\pi\frac{3}{2}r\cdot\frac{C}{r/2}\cdot\sup_{z\in\Gamma}|\varphi(z)| = 3C\sup_{z\in\Gamma}|(z-1)/z|$$

Note that $\sup_{z\in\Gamma}|\varphi(z)|$ is attained at the point $x = c - \frac{3}{2}r = \frac{1-3\delta/2}{1-\delta^2}$. Therefore

$$\sup_{z\in\Gamma}|\varphi(z)| = \frac{1-x}{x} = \delta\cdot\frac{3/2-\delta}{1-3\delta/2} \le \delta\cdot\frac{3/2}{1-3/4} = 6\delta.$$

Therefore $\|\varphi(T)\| \le 6C\delta$ and we are done. $\square$

4. Conjectures and open questions

In the conclusion of the paper let us state some conjectures. Let T be a contraction ($\|T\| \le 1$), and let $\sigma(T) \ne \overline{\mathbb{D}}$. Denote by T_μ the “Möbius transformation” of T,

$$T_\mu := (T-\mu I)(I-\overline{\mu}T)^{-1}, \qquad \mu\in\mathbb{D}.$$

Note that it $\|T\| \le 1$, then $\|T_\mu\| \le 1$, $\forall\mu\in\mathbb{D}$.

Let us recall that $\|A\|_{\mathfrak{S}_p}$ stands for the Schatten–von-Neumann norm of the operator A,

$$\|A\|_{\mathfrak{S}_p} = \Big(\sum_0^\infty s_n(A)^p \Big)^{1/p}.$$

It was shown above in Section 3, the Linear Resolvent Growth condition, as well as the similarity to a normal operator are invariant with respect to linear fractional transformations, in particular, with respect to the above "Möbius transformations". Since the "Möbius transformation" maps a contraction to contraction, the following conjecture seems plausible.

Conjecture 4.1. *If $\|T\| \le 1$, $\sigma(T) \ne \overline{\mathbb{D}}$, and*

$$\sup_{\mu \in \mathbb{D}} \|I - T_\mu^* T_\mu\|_{\mathfrak{S}_1} < \infty, \tag{4.1}$$

then the Linear Resolvent Growth condition (0.1) *implies that T is similar to a normal operator.*

We think that the trace class $\mathfrak{S}_1$ plays a critical role here.

Conjecture 4.2. *The condition* (4.1) *is sharp, i. e. given $p > 1$ one can find an operator T, $\|T\| \le 1$, $\sigma(T) \ne \overline{\mathbb{D}}$ satisfying the Linear Resolvent Growth condition* (0.1) *and*

$$\sup_{\mu \in \mathbb{D}} \|I - T_\mu^* T_\mu\|_{\mathfrak{S}_p} < \infty,$$

but not similar to a normal operator.

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