COMMON COMPLEMENTS OF TWO SUBSPACES OF A HILBERT SPACE

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Abstract. In this paper we find a necessary and sufficient condition for two closed subspaces, $\mathcal{X}$ and $\mathcal{Y}$, of a Hilbert space $\mathcal{H}$ to have a common complement, i.e. a subspace $\mathcal{Z}$ having trivial intersection with $\mathcal{X}$ and $\mathcal{Y}$ and such, that $\mathcal{H} = \mathcal{X} + \mathcal{Z} = \mathcal{Y} + \mathcal{Z}$.

Unlike the finite dimensional case the condition is significantly more subtle than simple equalities of dimensions and codimensions, and non-trivial examples of subspaces without a common complement are possible.

0. Introduction

0.1. **Statement of the problem and discussion.** In this paper we study when two subspaces $\mathcal{X}$ and $\mathcal{Y}$ of a Hilbert space $\mathcal{H}$ possess a common complement. Recall that a subspace $\mathcal{Z}$ of a Banach space $\mathcal{H}$ is called a *complement* of (or a complementary subspace to) a subspace $\mathcal{X} \subset \mathcal{H}$ if $\mathcal{X}$ and $\mathcal{Z}$ have trivial intersection and $\mathcal{H} = \mathcal{X} + \mathcal{Z}$. The latter means that any vector $h \in \mathcal{H}$ can be (uniquely, because $\mathcal{X} \cap \mathcal{Z} = \{0\}$) represented as $h = x + z$, $x \in \mathcal{X}$, $z \in \mathcal{Z}$. This unique representation can serve as an alternative definition of a complement.

Clearly, if $\mathcal{Z}$ is a complement of $\mathcal{X}$, then $\mathcal{X}$ is a complement of $\mathcal{Z}$, and sometimes we will call $\mathcal{X}$ and $\mathcal{Z}$ *complementary subspaces.*

Unlike the finite dimensional case ($\dim \mathcal{H} < \infty$) the conditions

(i) $\mathcal{X} \cap \mathcal{Z} = \{0\}$ and
(ii) $\mathcal{X} + \mathcal{Z}$ is dense in $\mathcal{H}$

are not sufficient for $\mathcal{X}$ and $\mathcal{Z}$ to be complementary subspaces: one more condition is needed. Namely, the closed graph theorem implies that if $\mathcal{Z}$ complements $\mathcal{X}$, then the skew projection $\mathcal{P} = \mathcal{P}_{\mathcal{X}||\mathcal{Z}}$ onto $\mathcal{X}$ parallel to $\mathcal{Z}$,

$$\mathcal{P}_{\mathcal{X}||\mathcal{Z}}(x + z) = x, \qquad x \in \mathcal{X}, z \in \mathcal{Z} \tag{0.1}$$

is a bounded operator. Under the above assumptions (i) and (ii) this condition is necessary and sufficient for the subspaces $\mathcal{X}$ and $\mathcal{Z}$ to be complementary subspaces.

In the finite dimensional finding a common complement is trivial. If $\mathcal{X} \subset \mathcal{H}$, and $\dim \mathcal{X} = n$, $\dim \mathcal{H} = N$, then the collection of all subspaces $\mathcal{Z}$ complementary to $\mathcal{X}$ is an open dense subset of the set of all subspaces of

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dimension $m = N - n$ (Grassmannian). Note, that the set of all subspaces complementary to $\mathcal{X}$ is a set of full measure in the above Grassmannian (which is a compact smooth manifold of dimension $n \times (N-n)$). So, using the Baire category theorem or measure theoretic reasoning one can conclude that any countable set of subspaces of dimension n has a common complement, and moreover, the set of all such common complements is a set of second category and a set of full measure in the Grassmannian of subspaces of dimension $m = N - n$.

The situation in the infinite dimensional case, as Theorem 0.1 below shows, is much more interesting. Of course, one could immediately see that the equality of dimensions (and codimensions) is not sufficient for the existence of a common complement. Indeed, it is possible that subspaces $\mathcal{X}$ and $\mathcal{Y}$ have equal dimensions and codimensions, but $\mathcal{X} \subsetneqq \mathcal{Y}$ so they do not have a common complement.

But the situation is, in fact, much more interesting. It can be easily shown, see Corollary 1.4 below, that the existence of a common complement implies that

$$\begin{aligned} \operatorname{codim}_{\mathcal{X}} \mathcal{X} \cap \mathcal{Y} := \dim(\mathcal{X} \ominus (\mathcal{X} \cap \mathcal{Y})) = \\ = \dim(\mathcal{Y} \ominus (\mathcal{X} \cap \mathcal{Y})) =: \operatorname{codim}_{\mathcal{Y}} \mathcal{X} \cap \mathcal{Y}, \end{aligned}$$

and we thought for some time that this equality of codimensions would be sufficient. To our surprise, this simple necessary condition turns out to be not sufficient, and the real necessary and sufficient condition is much more subtle.

However, in some "philosophical" sense the equality of codimensions is necessary and sufficient. Namely, it is necessary and sufficient if we replace the intersection $\mathcal{X} \cap \mathcal{Y}$ by the "ε-intersection", see Theorem 5.1 for the precise statement.

Also note that we do not even have a conjecture about when three subspaces have a common complement.

0.2. **Main result.** To state our main result let us recall how one can describe the geometry of a pair of subspaces up to unitary equivalence. Let $\mathbf{P} = \mathbf{P}_{\mathcal{Y}}$ be the orthogonal projection onto $\mathcal{Y}$, and let the operator $\mathbf{G} : \mathcal{X} \to \mathcal{Y}$ (Gramian) be defined by

$$\mathbf{G} = \mathbf{P}_{\mathcal{Y}} \big| \mathcal{X}.$$

Clearly the adjoint operator $\mathbf{G}^* : \mathcal{Y} \to \mathcal{X}$ is defined by $\mathbf{G}^* = \mathbf{P}_{\mathcal{X}} \big| \mathcal{Y}$.

It is a well known fact (and it will be shown later) that for any bounded operator G (from one Hilbert space to another) the essential parts of operators G^*G and GG^*, i.e. the operators $G^*G \big| (\ker G)^\perp$ are unitarily equivalent.

So, the geometry of a pair of subspaces is completely determined by the following objects:

1. The operator $\mathbf{G}^*\mathbf{G}$, or even only its essential part $\mathbf{G}^*\mathbf{G} \big| (\ker \mathbf{G})^\perp$;

2. Dimensions of two subspaces, $\mathcal{X}_0$ and $\mathcal{Y}_0$

$$\mathcal{X}_0 := \ker \mathbf{G} = \{x \in \mathcal{X} : x \perp \mathcal{Y}\}, \qquad \mathcal{Y}_0 := \ker \mathbf{G}^* = \{y \in \mathcal{Y} : x \perp \mathcal{X}\}$$

The following theorem is the main result of the paper.

Theorem 0.1. *Let $\mathcal{E}(\,.\,)$ be the spectral measure of the operator $\mathbf{G}^*\mathbf{G}$ (or of $\mathbf{G}^*\mathbf{G}\big|(\ker \mathbf{G})^\perp$). Then the subspaces $\mathcal{X}$ and $\mathcal{Y}$ have a common complement if and only if for*

$$\dim \mathcal{X}_0 + \dim \mathcal{E}((0, 1-\varepsilon))\mathcal{X} = \dim \mathcal{Y}_0 + \dim \mathcal{E}((0, 1-\varepsilon))\mathcal{X} \tag{0.2}$$

for some $\varepsilon > 0$ (for all sufficiently small $\varepsilon > 0$).

Remark 0.2. We do not assume that the space $\mathcal{H}$ is separable. The dimension in this case means the cardinality of an orthonormal basis (it is well known, see Section 1.1 below, that it doesn't depend on the choice of a basis). We add cardinalities according to usual rules, cf [2, Corollary I.4.30], i.e. the sum is the maximal dimension, except the case when both dimensions are finite.

Remark 0.3. It is easy to see that one can always replace $\mathcal{E}((0, 1-\varepsilon))$ by $\mathcal{E}((0, 1-\varepsilon])$ in the condition (0.2).

Remark 0.4. If $\dim \mathcal{X}_0 = \dim \mathcal{Y}_0$ then a common complement always exists.

Remark 0.5. If $\mathcal{H}$ is a separable space, the subspaces $\mathcal{X}$ and $\mathcal{Y}$ do not have a common complement if and only if $\dim \mathcal{X}_0 \neq \dim \mathcal{Y}_0$ and the operator $\mathbf{I} - \mathbf{G}^*\mathbf{G}\big|(\ker \mathbf{G})^\perp$ is compact.

Indeed, if there is no common complement, then $\dim \mathcal{X}_0 \neq \dim \mathcal{Y}_0$ and $\dim \mathcal{E}((0, 1-\varepsilon])\mathcal{X}$ is finite for all $\varepsilon > 0$. The latter exactly means that $I - \mathbf{G}^*\mathbf{G}$ is a compact operator.

1. Preliminaries

1.1. **Remarks about dimension.** This subsection deals with the definition of dimension for *non-separable* Hilbert spaces. A reader not interested in the non-separable case may skip this subsection, since the corresponding results are trivial for separable spaces.

To define the dimension we need to assume the Axiom of Choice, but it is usually assumed in functional analysis, for it is necessary for many results (Hahn–Banach theorem, existence of an orthonormal basis in the non-separable case).

The *dimension* of a Hilbert space (or subspace) is defined as the cardinality of an orthonormal basis. An old theorem due to Löwig and Rellich, see [1, Theorem IV.4.14] asserts that all orthonormal bases in a given Hilbert space H have the same cardinality, so the dimension is well defined.

Since unitary operators map orthonormal bases to orthonormal bases, two Hilbert spaces are isometrically isomorphic if and only if they have the same dimension.

If $A : H_1 \to H_2$ is an isomorphism (a bounded invertible operator) between two Hilbert spaces, it can be represented as $A = UR$ (polar decomposition), where $R = |A| := (A^*A)^{1/2}$ and $U : H_1 \to H_2$ is a unitary operator. So, if two Hilbert spaces are isomorphic, they are isometrically isomorphic, and therefore the dimension is preserved under the isomorphism.

1.2. **Codimension.** The *codimension* of a subspace X of a Hilbert space H is defined as $\dim X^\perp$. The proposition below is trivial for separable spaces, but it is also very easy to prove in general case.

Proposition 1.1. *Let X and Z be complementary subspaces of a Hilbert space H. Then* $\operatorname{codim} X = \dim Z$*.*

Proof. Take arbitrary $y \in X^\perp$. Since Z complements X, y has the unique decomposition

$$y = x + z, \quad x \in X, z \in Z, \qquad \|z\| \le C\|y\|.$$

Therefore, the orthogonal projection $\mathbf{P}_{X^\perp}$ onto $X^\perp$ maps Z isomorphically onto $X^\perp$, hence $\dim Z = \dim X^\perp$. □

Corollary 1.2. *The codimension of a subspace is preserved under isomorphisms (of the whole space).*

1.3. **Some trivial observations.** Before discussing the main result, let us make several observations. The following trivial proposition holds for arbitrary Banach spaces

Proposition 1.3. *The subspaces $\mathcal{X}$ and $\mathcal{Y}$ have a common complement if and only if there exists a bounded (not necessarily orthogonal) projection $\mathcal{P}$ onto one of the spaces (say, for definiteness onto $\mathcal{Y}$ such, that the operator*

$$\mathcal{G} : \mathcal{X} \to \mathcal{Y}, \qquad \mathcal{G} := \mathcal{P}\big|\mathcal{X}$$

is an isomorphism (bounded invertible operator) between $\mathcal{X}$ and $\mathcal{Y}$.

Proof. In such $\mathcal{P}$ exists, then $\mathcal{Z} := \ker \mathcal{P}$ is a common complement of $\mathcal{X}$ and $\mathcal{Y}$. Indeed, the projection $\mathcal{P}_{\mathcal{Y}||\mathcal{Z}} = \mathcal{P}$ is bounded, so $\mathcal{Z}$ is a complement of $\mathcal{Y}$. The projection $\mathcal{P}_{\mathcal{X}||\mathcal{Z}}$ onto $\mathcal{X}$ parallel to $\mathcal{Z}$ can be defined by

$$\mathcal{P}_{\mathcal{X}||\mathcal{Z}} = \mathcal{G}^{-1}\mathcal{P},$$

so it is also bounded. So $\mathcal{Z}$ is a complement of $\mathcal{X}$ as well. □

Now let us return to Hilbert spaces.

Corollary 1.4. *If subspaces $\mathcal{X}$ and $\mathcal{Y}$ of a Hilbert space have a common complement, then the dimensions of the spaces $\mathcal{X} \ominus (\mathcal{X} \cap \mathcal{Y})$ and $\mathcal{Y} \ominus (\mathcal{X} \cap \mathcal{Y})$ coincide (i.e.* $\operatorname{codim}_{\mathcal{X}} \mathcal{X} \cap \mathcal{Y} = \operatorname{codim}_{\mathcal{Y}} \mathcal{X} \cap \mathcal{Y}$*, where* $\operatorname{codim}_{\mathcal{X}}$ *stands for the codimension in $\mathcal{X}$).*

Proof. Trivial, since the operator $\mathcal{G}$ from the above Proposition 1.3 maps $\mathcal{X} \ominus (\mathcal{X} \cap \mathcal{Y})$ isomorphically onto $\mathcal{Y} \ominus (\mathcal{X} \cap \mathcal{Y})$ □

Proposition 1.5. *If $\mathcal{X}$, $\mathcal{Y} \subset \mathcal{H}$ have a common complement in the closure of $\mathcal{X}+\mathcal{Y}$ then $\mathcal{X}$ and $\mathcal{Y}$ have a common complement in $\mathcal{H}$.*

Proof. Let $\mathcal{Z}$ be the common complement of $\mathcal{X}$ and $\mathcal{Y}$ in $\operatorname{Clos}(\mathcal{X}+\mathcal{Y})$. Then $\mathcal{Z} \oplus (\mathcal{X}+\mathcal{Y})^\perp$ is a common complement of $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{H}$ □

Thus without loss of generality we may always assume $\operatorname{Clos}(\mathcal{X}+\mathcal{Y}) = \mathcal{H}$.

2. Sufficiency

In this section we prove that condition (0.2) is sufficient for the subspaces $\mathcal{X}$ and $\mathcal{Y}$ to have a common complement. We first treat several simple cases, and then we show that the general case can be treated as a "direct sum" of the simple cases.

Recall that the Gramian $\mathbf{G} : \mathcal{X} \to \mathcal{Y}$ is defined as

$$\mathbf{G} = \mathbf{P}_{\mathcal{Y}} | \mathcal{X}$$

and its adjoint $\mathbf{G}^*$ is defined by

$$\mathbf{G}^* = \mathbf{P}_{\mathcal{X}} | \mathcal{Y}.$$

2.1. **Some simple cases.** First we consider the case where $\mathcal{X}$ and $\mathcal{Y}$ are, in some sense, completely non-orthogonal.

Proposition 2.1. *If $\mathbf{G}$ is invertible then $\mathcal{Z} = \mathcal{Y}^\perp$ is a common complement of $\mathcal{X}$ and $\mathcal{Y}$.*

Proof. Follows immediately from Proposition 1.3 □

The next case will be treated by changing the inner product in $\mathcal{H}$. Note that having a common complement is a topological property, meaning it does not change if we replace the norm (inner product) in $\mathcal{H}$ by an equivalent one. If we have a Hilbert space H and $A = A^* \geq 0$ is a bounded and invertible operator in H, then $(\cdot,\cdot)_A$, $(f,g)_A := (A, f, g)$ defines an equivalent inner product in H (in fact, all equivalent inner products in H can be defined this way, but we won't need that in what follows).

Let $H = \mathcal{X} \oplus \mathcal{Y}$ be the orthogonal sum of $\mathcal{X}$ and $\mathcal{Y}$. Given an operator $G : \mathcal{X} \to \mathcal{Y}$, $\|G\| < 1$ consider a norm on H defined by the operator

$$A = A_G = \begin{pmatrix} I & G^* \\ G & I \end{pmatrix}$$

Clearly, if $\|G\| < 1$, the operator A_G is invertible, therefore the corresponding norm is equivalent to the original norm on H. Thus, all such norms are equivalent to each other.

Note that if $G = \mathbf{G}$, then the corresponding norm in H is the original norm in $\mathcal{X}+\mathcal{Y} \subset \mathcal{H}$. Also note that if $G = 0$ then $\mathcal{X} \perp \mathcal{Y}$ in the norm generated by A_G.

Proposition 2.2. *If $\|\mathbf{G}\| < 1$ and $\dim\{\mathcal{X} \ominus (\mathcal{X}\cap\mathcal{Y})\} = \dim\{\mathcal{Y} \ominus (\mathcal{X}\cap\mathcal{Y})\}$, then $\mathcal{X}$ and $\mathcal{Y}$ have a common complement.*

Proof. As it was said above in Proposition 1.5 we can assume without loss of generality that $\mathcal{X} + \mathcal{Y}$ is dense in $\mathcal{H}$.

The equality of dimensions imply that there exist an isomorphism (bounded invertible operator) $G : \mathcal{X} \to \mathcal{Y}$. Multiplying it by a small number we can always assume that $\|G\| < 1$. So, as we just discussed above, the norms generated by the operators A_G and $A_{\mathbf{G}}$ are equivalent, and both are equivalent to the norm corresponding to A_0 (meaning A_G with $G = 0$).

The norm corresponding to $A_{\mathbf{G}}$ is the norm on $\mathcal{X} + \mathcal{Y}$ inherited from $\mathcal{H}$. This norm is equivalent to the norm generated by A_0, therefore the subspace $\mathcal{X} + \mathcal{Y}$ is closed, and so $\mathcal{X} + \mathcal{Y} = \mathcal{H}$.

Therefore A_G gives the equivalent norm on $\mathcal{H}$. Note that in this norm the corresponding Gramian $\mathbf{G}$ equals G. Since it is invertible, Proposition 2.1 implies that $\mathcal{X}$ and $\mathcal{Y}$ have a common complement. $\square$

2.2. **The general case.** To treat the general case we split the subspaces $\mathcal{X}$ and $\mathcal{Y}$ into orthogonal sums, $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$, so that the subspaces $\mathcal{H}_k := \operatorname{Clos}(\mathcal{X}_k + \mathcal{Y}_k)$ are also orthogonal.

Then, if each pair $\mathcal{X}_k$, $\mathcal{Y}_k$ has a common complement in $\mathcal{H}_k$, $k = 1, 2$, then clearly $\mathcal{X}$ and $\mathcal{Y}$ have a common complement.

Let $\mathcal{E}(\cdot)$ denote the spectral measure of the operator $\mathbf{G}^*\mathbf{G}$ and $\mathcal{E}_*(\cdot)$ be the spectral measure of $\mathbf{G}\mathbf{G}^*$. Fix $a \in (0, 1)$ and define

$$\mathcal{X}_1 := \mathcal{E}([0, a))\mathcal{X}, \quad \mathcal{X}_2 := \mathcal{E}([a, 1])\mathcal{X}, \tag{2.1}$$

and similarly,

$$\mathcal{Y}_1 := \mathcal{E}_*([0, a))\mathcal{Y}, \quad \mathcal{Y}_2 := \mathcal{E}_*([a, 1])\mathcal{Y}. \tag{2.2}$$

where $a = 1 - \varepsilon$, ε is from the assumption (0.2) of the theorem.

2.2.1. *The case of trivial kernels.* Let us first consider the case when both $\ker \mathbf{G}$ and $\ker \mathbf{G}^*$ are trivial (then the assumption (0.2) is automatically satisfied for all $\varepsilon \in (0, 1)$).

Consider the polar decomposition $\mathbf{G} = UR$, where $R = (\mathbf{G}^*\mathbf{G})^{1/2}$ and $U : \mathcal{X} \to \mathcal{Y}$ is a unitary operator. Since $\mathbf{G}\mathbf{G}^* = UR^2U^* = U(\mathbf{G}^*\mathbf{G})U^*$ we have for the spectral measures

$$\mathcal{E}_* = U\mathcal{E}U^*.$$

This implies that $\mathcal{Y}_k = U\mathcal{X}_k$, $k = 1, 2$, and therefore $\dim \mathcal{X}_k = \dim \mathcal{Y}_k$.

Since $\mathcal{X}_k$ are $\mathbf{G}^*\mathbf{G}$-invariant, and so R-invariant

$$\mathbf{G}\mathcal{X}_k = UR\mathcal{X}_k \subset U\mathcal{X}_k = \mathcal{Y}_k,$$

and similarly $\mathbf{G}^*\mathcal{Y}_k \subset \mathcal{X}_k$. Note that it is also easy to prove that $\mathbf{G}\mathcal{X}_2 = \mathcal{Y}_2$ and $\mathbf{G}\mathcal{X}_1$ is dense in $\mathcal{Y}_1$, and similarly for $\mathcal{Y}_k$, but we won't need these facts now.

To show that $\mathcal{H}_1 \perp \mathcal{H}_2$ it is sufficient to show that $\mathcal{X}_1 \perp \mathcal{Y}_2$ and $\mathcal{X}_2 \perp \mathcal{Y}_1$. Let us show that $\mathcal{X}_1 \perp \mathcal{Y}_2$. Take $x \in \mathcal{X}_1$, $y \in \mathcal{Y}_2$. We have

$$(x, y) = (\mathbf{P}_{\mathcal{Y}} x, y) = (\mathbf{G}x, y) = 0$$

since $\mathbf{G}x \in \mathcal{Y}_1 \perp \mathcal{Y}_2$. The orthogonality $\mathcal{X}_2 \perp \mathcal{Y}_1$ is proved similarly.

Now we need to prove that the pairs $\mathcal{X}_k$, $\mathcal{Y}_k$, $k = 1, 2$ have a common complement. For the pair $\mathcal{X}_k$, $\mathcal{Y}_k$ the corresponding Gramian is the restriction $\mathbf{G}\big|\mathcal{X}_k$. Then for $\mathcal{X}_2$, $\mathcal{Y}_2$ the Gramian is invertible, and for $\mathcal{X}_1$, $\mathcal{Y}_1$ its norm is less than 1. Since, as we already discussed above, $\dim \mathcal{X}_k = \dim \mathcal{Y}_k$, Proposition 2.1 implies that subspaces $\mathcal{X}_2$, $\mathcal{Y}_2$ have a common complement, and Proposition 2.2 asserts the existence of a common complement for the pair $\mathcal{X}_1$, $\mathcal{Y}_1$. □

2.2.2. *The case of non-trivial kernels.* Let us now consider the general case, when we allow non-trivial kernels for $\mathbf{G}$ and $\mathbf{G}^*$. We set $a = 1 - \varepsilon$, where ε is from the assumption (0.2) of Theorem 0.1, and define $\mathcal{X}_k$, $\mathcal{Y}_k$ as above.

The simplest way to understand the geometry here, is to think that we first had the case of trivial kernels, and then we added to $\mathcal{X}_1$ and $\mathcal{Y}_1$ orthogonal (to everything else) subspaces $\ker \mathbf{G}$ and $\ker \mathbf{G}^*$ respectively. Since we added orthogonal subspaces, the orthogonality condition remains true. The subspaces $\mathcal{X}_2$, $\mathcal{Y}_2$ will not change, so this pair has a common complement. As for the pair $\mathcal{X}_1$, $\mathcal{Y}_1$, the norm of corresponding Gramian remains the same (by adding two orthogonal to everything subspaces, we just added zero blocks to the "old" Gramian), so it is less than 1. Assumption (0.2) of the theorem means that the dimensions of "new" $\mathcal{X}_1$ and $\mathcal{Y}_1$ coincide, so Proposition 2.2 implies that there is a common complement for this pair as well.

To write the last paragraph formally, let $\mathcal{X}^0 := \mathcal{X} \ominus \ker \mathbf{G}$, $\mathcal{Y}^0 := \mathcal{Y} \ominus \ker \mathbf{G}^*$, and let $\mathbf{G}_0 : \mathcal{X}^0 \to \mathcal{Y}^0$ be the restriction of $\mathbf{G}$. Let us denote by $\mathcal{X}_k^0$, $\mathcal{Y}_k^0$, $\mathcal{H}_k^0$ the corresponding subspaces for $\mathbf{G}_0$, and by $\mathcal{E}^0$, $\mathcal{E}_*^0$ the spectral measures for $\mathbf{G}_0$ and $\mathbf{G}_0^*$ respectively. Clearly

$$\mathcal{X}_2 = \mathcal{X}_2^0, \qquad \mathcal{Y}_2 = \mathcal{Y}_2^0,$$

and

$$\mathcal{X}_1 = \mathcal{X}_1^0 \oplus \ker \mathbf{G}, \qquad \mathcal{Y}_1 = \mathcal{Y}_1^0 \oplus \ker \mathbf{G}^*.$$

Since $\ker \mathbf{G} \perp \mathcal{Y} \oplus \mathcal{X}^0$, and $\ker \mathbf{G}^* \perp \mathcal{X} \oplus \mathcal{Y}^0$, the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are orthogonal.

Because they coincide with $\mathcal{X}_2^0$ and $\mathcal{Y}_2^0$, the subspaces $\mathcal{X}_2$, $\mathcal{Y}_2$ possess a common complement. Since, as we already discussed for the case of trivial kernels

$$\dim \mathcal{X}_1^0 = \dim \mathcal{Y}_1^0 = \dim \mathcal{E}^0((0, a))\mathcal{X}^0 = \dim \mathcal{E}((0, a))\mathcal{X},$$

the assumption (0.2) implies

$$\dim \mathcal{X}_1 = \dim \mathcal{X}_1^0 + \dim \ker \mathbf{G} = \dim \mathcal{Y}_1^0 + \dim \ker \mathbf{G}^* = \dim \mathcal{Y}_1.$$

Now notice that $\|\mathbf{G}\big|\mathcal{X}_1\| = \|\mathbf{G}_0\big|\mathcal{X}_1^0\| < 1$ (the operators differ by zero blocks), so Proposition 2.2 implies that $\mathcal{X}_1$ and $\mathcal{Y}_1$ have a common complement.

3. Necessity

Lemma 3.1. *Let X and Y be subspaces of a Hilbert space H. Suppose there exists an isomorphism (bounded invertible operator) $A : X \to Y$ such that*

$$\|x - Ax\| \le q\|x\| \qquad \forall x \in X,$$

for some $q < 1$. Then $\operatorname{codim} X = \operatorname{codim} Y$.

Proof. Define an operator $A_X : X \to X$ by $A_X x := \mathbf{P}_X Ax$, where $\mathbf{P}_X$ is orthogonal projection onto X. Since

$$\|A_X x - x\| = \|\mathbf{P}_X(Ax - x)\| \le \|Ax - x\| \le q\|x\|,$$

we have $\|A_X - \mathbf{I}\| \le q < 1$, so A_X is invertible.

For $y \in Y$ we have

$$\mathbf{P}_X y = A_X A^{-1} y,$$

so $\mathbf{P}_X$ defines an isomorphism between X and Y. Therefore, by Proposition 2.1, $X^\perp$ is a common complement for X and Y. Hence, see Proposition 1.1, $\operatorname{codim} X = \operatorname{codim} Y = \dim X^\perp$. $\square$

Now suppose that $\mathcal{X}$ and $\mathcal{Y}$ have a common complement. Then, by Proposition 1.3, there exists a bounded projection $\mathcal{P}$ onto $\mathcal{Y}$ such that $\mathcal{G} := \mathcal{P}\big|\mathcal{X}$ is an isomorphism between $\mathcal{X}$ and $\mathcal{Y}$.

We want to prove that condition (0.2) from Theorem 0.1 holds for some $\varepsilon > 0$. In the notation of the previous section, see (2.1) (2.2), this condition can be rewritten as

$$\operatorname{codim}_{\mathcal{X}} \mathcal{X}_2 := \dim(\mathcal{X} \ominus \mathcal{X}_2) = \dim(\mathcal{Y} \ominus \mathcal{Y}_2) =: \operatorname{codim}_{\mathcal{Y}} \mathcal{Y}_2$$

where $a = 1 - \varepsilon$. Here $\operatorname{codim}_{\mathcal{X}}$ stands for the codimension in $\mathcal{X}$. Since $\mathcal{G}$ is an isomorphism between $\mathcal{X}$ and $\mathcal{Y}$,

$$\operatorname{codim}_{\mathcal{X}} \mathcal{X}_2 = \operatorname{codim}_{\mathcal{Y}} \mathcal{G}\mathcal{X}_2.$$

So we want to show that the subspaces $\mathcal{G}\mathcal{X}_2 = \mathcal{P}\mathcal{X}_2$ and $\mathcal{Y}_2 = \mathbf{G}\mathcal{X}_2 = \mathbf{P}_{\mathcal{Y}}\mathcal{X}_2$ have the same codimension in $\mathcal{Y}$. To do that we will use Lemma 3.1 above.

Take $x \in \mathcal{X}_2$. Then $\|\mathbf{G}x\|^2 \ge a\|x\|^2$, so

$$\|x\|^2 - \|\mathbf{G}x\|^2 \le (1 - a)\|x\|^2 \le \frac{1-a}{a}\|\mathbf{G}x\|^2 = \frac{\varepsilon}{1-\varepsilon}\|\mathbf{G}x\|^2.$$

Hence

$$\begin{aligned}
\|\mathbf{G}x - \mathcal{P}x\|^2 &= \|\mathbf{P}_{\mathcal{Y}}x - \mathcal{P}x\|^2 = \|\mathcal{P}(\mathbf{P}_{\mathcal{Y}}x - x)\|^2 \le \|\mathcal{P}\|^2 \cdot \|\mathbf{P}_{\mathcal{Y}}x - x\|^2 \\
&= \|\mathcal{P}\|^2(\|x\|^2 - \|\mathbf{P}_{\mathcal{Y}}x\|^2) = \|\mathcal{P}\|^2(\|x\|^2 - \|\mathbf{G}x\|^2) \\
&\le \frac{\varepsilon}{1-\varepsilon}\|\mathcal{P}\|^2\|\mathbf{G}x\|^2.
\end{aligned}$$

Thus the assumptions of Lemma 3.1 hold for subspaces $\mathbf{P}_{\mathcal{Y}}\mathcal{X}_2$ and $\mathcal{P}\mathcal{X}_2$ in $\mathcal{Y}$ with the operator $A : \mathbf{P}_{\mathcal{Y}}\mathcal{X}_2 \to \mathcal{P}\mathcal{X}_2$ defined by

$$A = \mathcal{P} \cdot (\mathbf{G}\big|\mathcal{X}_2)^{-1}$$

(recall that $\mathbf{G}$ maps $\mathcal{X}_2$ isomorphically to $\mathcal{Y}_2$).

4. An Example of subspaces without common complement

As we said above, if the space $\mathcal{H}$ is separable the only situation when a common complement does not exists is when $\mathbf{I}-\mathbf{G}^*\mathbf{G}\big|(\ker\mathbf{G})^\perp$ is a compact operator, and $\dim\ker\mathbf{G} \neq \dim\ker\mathbf{G}^*$. This gives us the possibility to construct non-trivial examples of subspaces without a common complement.

By non-trivial example we mean here a pair of subspaces satisfying the simple necessary condition

$$\operatorname{codim}_{\mathcal{X}}(\mathcal{X}\cap\mathcal{Y}) = \operatorname{codim}_{\mathcal{Y}}(\mathcal{X}\cap\mathcal{Y}),$$

and not possessing a common complement.

In this section we construct subspaces $\mathcal{X}$ and $\mathcal{Y}$ of equal dimensions and codimensions and with trivial intersection, which do not have a common complement.

Let $\mathcal{H} = \ell^2$ the space of square summable sequences with indices $0, 1, 2, \ldots$. We let $\{e_k\}_{k=0}^\infty$ be the standard orthonormal basis of $\mathcal{H}$. We will define $\mathcal{X}$ and $\mathcal{Y}$ by defining a basis for these subspaces. We define:

$$y_0 = e_0,$$

and for $k \geq 1$

$$y_k = \cos\frac{1}{k}e_{2k-1} + \sin\frac{1}{k}e_{2k}$$
$$x_k = \cos\frac{1}{k}e_{2k-1} - \sin\frac{1}{k}e_{2k}.$$

We see that $\mathcal{X} = \operatorname{span}\{x_i\}_{i=1}^\infty$ and $\mathcal{Y} = \operatorname{span}\{y_i\}_{i=0}^\infty$ are subspaces of equal dimension and codimension, $\mathcal{X}\cap\mathcal{Y} = \{0\}$, with no common complement. Indeed, it is trivial that $\mathbf{G}$ is a compact perturbation of $\mathbf{I}$, and $0 = \dim\ker\mathbf{G} \neq \dim\ker\mathbf{G}^* = 1$.

5. A geometric interpretation of the results.

The interesting thing about the main result of the paper is that despite the fact that the existence of a common complement is a topological condition (i.e. it does not change when one replace the norm by an equivalent one), the orthogonality mysteriously appears in the results and plays the important rôle here.

In this section we give a more geometric version of the above Theorem 0.1, which does not include orthogonality explicitly. We say here "explicitly" because it still requires a Hilbert space norm. We do not know if it is true if one replaces the Hilbert space norm by an equivalent Banach space norm.

Let us introduce a few definitions. Let K be a subset of a Hilbert space X. We define the *lower linear dimension* of K as

$$\sup\{\dim L : L \text{ is a linear subspace of } K\}$$

and the *upper linear codimension* of K by

$$\inf\{\operatorname{codim} L : L \text{ is a linear subspace of } K\}.$$

If the reader is not comfortable with taking supremum or infimum of a family of cardinalities, he should not be worried, since in our case there always be subspaces of maximal dimension and subspaces of minimal codimension.

Let $\varepsilon > 0$ For the subspaces $\mathcal{X}$, $\mathcal{Y}$ of $\mathcal{H}$ define the cones

$$\mathcal{K}^\varepsilon_{\mathcal{X}} := \{x \in \mathcal{X} : \operatorname{dist}(x, \mathcal{Y}) \le \varepsilon \|x\|\},$$
$$\mathcal{K}^\varepsilon_{\mathcal{Y}} := \{y \in \mathcal{Y} : \operatorname{dist}(y, \mathcal{X}) \le \varepsilon \|y\|\}.$$

For small ε one can treat the cones $\mathcal{K}^\varepsilon_{\mathcal{X}}$, $\mathcal{K}^\varepsilon_{\mathcal{Y}}$ as "ε-intersection" of $\mathcal{X}$ and $\mathcal{Y}$.

As it was said before in Corollary 1.4 if the subspaces $\mathcal{X}$ and $\mathcal{Y}$ have a common complement, the intersection $\mathcal{X} \cap \mathcal{Y}$ has the same codimensions in $\mathcal{X}$ and in $\mathcal{Y}$. Theorem 0.1 shows that the equality of codimensions is not sufficient for the existence of a common complement. The theorem below, which is a reformulation of the main result (Theorem 0.1) shows that if one replaces the intersection $\mathcal{X} \cap \mathcal{Y}$ with the "ε-intersections" $\mathcal{K}^\varepsilon_{\mathcal{X}}$, $\mathcal{K}^\varepsilon_{\mathcal{Y}}$, then the equality of codimensions is necessary and sufficient.

Theorem 5.1. *Subspaces $\mathcal{X}$ and $\mathcal{Y}$ of a Hilbert space $\mathcal{H}$ have a common complement if and only if for some (small) $\varepsilon > 0$ the* upper linear codimensions *of the cones $\mathcal{K}^\varepsilon_{\mathcal{X}}$ in $\mathcal{X}$ and $\mathcal{K}^\varepsilon_{\mathcal{Y}}$ in $\mathcal{Y}$ coincide.*

Proof. To prove the theorem we will show that the upper linear codimensions of the cones $\mathcal{K}^\varepsilon_{\mathcal{X}}$ and $\mathcal{K}^\varepsilon_{\mathcal{Y}}$ equal to

$$\dim \mathcal{E}([0, 1 - \varepsilon^2)] = \dim \ker \mathbf{G} + \dim \mathcal{E}((0, 1 - \varepsilon^2))$$

and

$$\dim \mathcal{E}_*([0, 1 - \varepsilon^2)] = \dim \ker \mathbf{G}^* + \dim \mathcal{E}_*((0, 1 - \varepsilon^2))$$

respectively. Recalling that $\dim \mathcal{E}_*((0, 1 - \varepsilon^2)) = \dim \mathcal{E}((0, 1 - \varepsilon^2))$ we will see that the condition of Theorem 5.1 is exactly the assumption (0.2) of Theorem 0.1 (with ε replaced by ε^2).

To compute the codimension of the cones, let us notice that for $x \in \mathcal{X}$

$$\big[\operatorname{dist}(x, \mathcal{Y})\big]^2 = \|x\|^2 - \|\mathbf{P}_{\mathcal{Y}} x\|^2 = \|x\|^2 - \|\mathbf{G}x\|^2.$$

Therefore, the cone $\mathcal{K}^\varepsilon_{\mathcal{X}}$ is the cone of nonnegative vectors of the operator $A = \mathbf{G}^*\mathbf{G} - (1 - \varepsilon^2)\mathbf{I}$, i.e. $\mathcal{K}^\varepsilon_{\mathcal{X}} = \{x \in \mathcal{X} : (Ax, x) \ge 0\}$, and similarly, $\mathcal{K}^\varepsilon_{\mathcal{Y}}$ is the cone of the nonnegative vectors of $A_* := \mathbf{G}\mathbf{G}^* - (1 - \varepsilon^2)\mathbf{I}$. Then the theorem follows from the lemma below.

Let A be a bounded selfadjoint operator in a Hilbert space H, and let K be the cone of nonnegative vectors of A, $K := \{x \in H : (Ax, x) \ge 0\}$. Let $\mathcal{E}$ be the spectral measure of A, and let

$$H_+ := \mathcal{E}([0, \infty)), \qquad H_- := \mathcal{E}((-\infty, 0)).$$

Lemma 5.2. *The upper linear codimension of the cone K is exactly*

$$\operatorname{codim} H_+ = \dim H_-.$$

The theorem follows immediately from the lemma, since for the operator $A = \mathbf{G}^*\mathbf{G} - (1-\varepsilon^2)\mathbf{I}$ defined above,

$$H_+ = \mathcal{E}([1-\varepsilon^2, 1]), \qquad H_- = \mathcal{E}((0, 1-\varepsilon^2)),$$

and similarly for A_*.

Proof of Lemma 5.2. Consider a (closed) subspace $X \subset K$. Let P be the orthogonal projection onto H_+, and let $Y := \operatorname{Clos} PX$.

By the construction of Y, the set $Y_0 := \{y \in Y : y \perp X\} = \{0\}$ Since $X \perp H_+ \ominus Y$, any vector $x \in X$ orthogonal to Y is automatically orthogonal to H_+. But since $X \subset K$, we have $X \cap H_- = \{0\}$, so $X_0 := \{x \in X : x \perp Y\} = \{0\}$. Therefore the condition (0.2) of Theorem 0.1 are satisfied (see Remark 0.4 there), and X and Y have a common complement, say Z. Proposition 1.1 implies that

$$\operatorname{codim} X = \operatorname{codim} Y = \dim Z,$$

and since $Y \subset H_+$, we have $\operatorname{codim} Y \geq \operatorname{codim} H_+$. Thus, H_+ has the smallest codimension among all subspaces of K. □

□

Remark 5.3. As we had said above in the beginning of this section, despite the fact that the existence of a common complement is a topological condition, the Hilbert space structure (the inner product, orthogonality) mysteriously appears in the results. The condition of Theorem 5.1 does not involve any Hilbert space structure, but we still need it in the proof. We do not know if the theorem is still true if we replace the Hilbert space norm by a Banach norm (not generated by a scalar product).

Remark 5.4. The condition of Theorem 5.1 is much harder to check then condition (0.2) of Theorem 0.1. However we think that Theorem 5.1 is still interesting, since its assumption can be considered as a generalization of a simple (and geometric) necessary condition

$$\operatorname{codim}_{\mathcal{X}} \mathcal{X} \cap \mathcal{Y} = \operatorname{codim}_{\mathcal{Y}} \mathcal{X} \cap \mathcal{Y},$$

see Corollary 1.4 in Section 1 above.

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