ESTIMATES IN THE CORONA THEOREM AND IDEALS OF H^∞: A PROBLEM OF T. WOLFF

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The main result of the paper is that there exist functions $f_1, f_2, f \in H^\infty$ satisfying the "Corona condition"

$$|f_1(z)| + |f_2(z)| \geq |f(z)| \qquad \forall z \in \mathbb{D},$$

and such, that f^2 does not belong to the ideal $\mathcal{I}$ generated by f_1, f_2, i. e. f^2 cannot be represented as $f^2 \equiv f_1 g_1 + f_2 g_2$, $g_1, g_2 \in H^\infty$. This gives a negative answer to an old question by T. Wolff.

Note, that it was well known before that under the same assumptions f^p belongs to the ideal if $p > 2$, but a counterexample can be constructed for $p < 2$, so our case $p = 2$ is a critical one.

To get the main result we improved lower estimates for the solution of the Corona problem. Namely, we proved that given $\delta > 0$ there exist finite Blaschke products f_1, f_2 satisfying the Corona condition

$$|f_1(z)| + |f_2(z)| \geq \delta \qquad \forall z \in \mathbb{D},$$

and such, that for any $g_1, g_2 \in H^\infty$ satisfying $f_1 g_1 + f_2 g_2 \equiv 1$ (solution of the Corona problem), the estimate $\|g_1\|_\infty \geq C\delta^{-2} \log(-\log \delta)$ holds. The estimate $\|g_1\|_\infty \geq C\delta^{-2}$ was obtained earlier by V. Tolokonnikov.