THE GAP BETWEEN COMPLEX STRUCTURED SINGULAR VALUE μ AND ITS UPPER BOUND IS INFINITE

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0. Introduction

The (complex) structured singular value $\mu = \mu(A)$ of a square matrix A was introduced in [6] and is defined as follows

$$\begin{aligned}\mu(A) :&= \Big(\inf\{\|\Delta\| \,:\, \Delta \text{ is diagonal, } I - M\Delta \text{ is not invertible}\}\Big)^{-1} \\ &= \Big(\inf\{\|\Delta\| \,:\, \Delta \text{ is diagonal, } I - \Delta M \text{ is not invertible}\}\Big)^{-1}\end{aligned}$$

Here and below, $\|A\|$ always denotes the induced (by the Euclidian norm in $\mathbb{C}^n$) operator norm of A, i. e. its maximal singular value. Entries of the matrices A and Δ are complex, so we deal with *complex* structured singular value.

The first equality above is the definition of μ, the second is a simple exercise in linear algebra.

The structured singular value μ was introduced in connection with robust control with structured uncertainties. Consider, for example, the system on Fig. 1 with uncertainty Δ in the feedback loop. Here G is causal stable LTI (Linear Time Invariant) plant.

An important notion in robust stability is so called *stability margin*. Let our uncertainty Δ belongs to some class $\mathbb{U}$ of stable causal systems. The *stability margin*

$$\begin{aligned}\rho_{\mathbb{U}}(G) :&= \inf\{\|\Delta\| \,:\, \text{the system on Fig. 1 is not stable}\} \\ &= \inf\{\|\Delta\| \,:\, I - G\Delta \text{ is not invertible}\}\end{aligned}$$

(let us remind the reader that the system on Fig. 1 with stable G and Δ is stable if and only if $I - G\Delta$ is invertible).

The first result in robust stability is the so called *Small Gain Theorem*, which states (under some technical assumptions) that if the class of uncertainties $\mathbb{U}$ coincides with all causal LTI (or Linear Time Varying, or Nonlinear Time Invariant, or Nonlinear Time Varying) systems, then $\rho_{\mathbb{U}}(G) = 1/\|G\|$.

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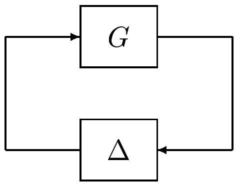

FIGURE 1. Uncertainty Δ in the feedback loop

One can consider different classes $\mathbb{U}$ of uncertainties. Define the *structured norm* $\mathrm{SN}_{\mathbb{U}}(G) := 1/\rho_{\mathbb{U}}(G)$; the name structured norm is clear from the Small Gain Theorem.

Very often uncertainty Δ has some additional structure. In this paper we are interested in the case when G is causal LTI system with n inputs and n outputs, i. e. the multiplication by a (square, $n \times n$) H^∞ matrix-function on the vector-valued H^2 space, and Δ is a diagonal LTI plant, i. e. a multiplication by a *diagonal* $n \times n$ H^∞ matrix-function. This means that we have independent uncertainties in each output channel.

Such situations appear naturally in robust control. For example, the problem of robust stabilization in the presence of independent LTI uncertainties in measurements and control channels, see Fig. 2, can be reduced to the situation in Fig. 1 with $\Delta = \mathrm{diag}\{\Delta_1, \Delta_2\}$.

It is a well known result, see [6], that if the class $\mathbb{U}$ of uncertainties consists of (causal) diagonal LTI systems, then

$$\mathrm{SN}_{\mathbb{U}}(G) = \sup_{\omega \in \mathbb{R}} \mu(\hat{G}(i\omega)), \tag{0.1}$$

where $\hat{G}$ is the transfer function of the plant G. (The corresponding stability margin equals $1/\mathrm{SN}_{\mathbb{U}}(G)$.)

One can consider uncertainties of more complicated structure, for example block diagonal matrices with fixed sizes of blocks, cf [6], and introduce corresponding μ. The formula (0.1) above is true in this situation as well.

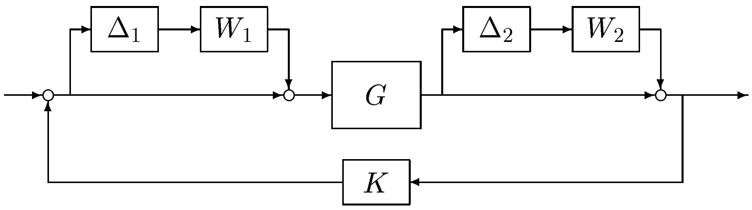

FIGURE 2. Feedback stabilization with uncertainties in observation and control channels.

Note, that although transfer functions $\hat{G}$, $\hat{\Delta}$ are *real* rational matrix functions (i. e. their entries are rational functions with real coefficients), the values $\hat{G}(i\omega)$, $\hat{\Delta}(i\omega)$ are complex matrices, so in the definition of μ one has to consider complex matrices.

So, the structured singular value μ is an important characteristic of a matrix. Unfortunately, it is very hard to compute: the computation of μ reduces to a non-convex optimization problem.

The standard way to cope with this difficulty, is to introduce some (easy computable) upper bound for μ.

Clearly $\mu(A) \leqslant \|A\|$, so $\|A\|$ is a trivial (and easily computable) upper bound. Unfortunately, this upper bound is too conservative: using it brings us back in the situation of the Small Gain Theorem. And μ was introduced exactly to improve the Small Gain Theorem by taking into account the structure of uncertainty.

Also, it is very easy to construct an example of a (2×2) matrix A , such that the ratio $\|A\|/\mu(A)$ is as large as one wants.

The following quantity gives a closer (and widely used) upper bound for μ. Let D be an invertible diagonal matrix. Then for diagonal Δ

$$I - DAD^{-1}\Delta = D(I - A\Delta)D^{-1},$$

and so $\mu(A) = \mu(DAD^{-1}) \leqslant \|DAD^{-1}\|$.

Therefore

$$\overline{\mu}(A) := \inf\{\|DAD^{-1}\| \,:\, D \text{ is diadonal and invertible}\},$$

is an upper bound for μ, $\mu(A) \leqslant \overline{\mu}(A)$.

The upper bound $\overline{\mu}$ is also easy to compute: the computation reduces to a convex optimization problem (solving Linear Matrix Inequalities (LMI)).

This $\overline{\mu}$ is much better upper bound. First of all, it was proved by Doyle [1], see also [6, Section 9], that $\overline{\mu}$ coincide with μ for $n \leqslant 3$ (in general (block diagonal) case n is the number of diagonal blocks in Δ; in our case it is just the size of the matrix A)

It is known that for $n > 3$ the upper bound is a conservative estimate: it is not difficult to find a matrix A such that $\mu(A) < \overline{\mu}(A)$, see [6]. However, in all numerical experiments the ratio $\overline{\mu}(A)/\mu(A)$ was never too big, so it was conjectured that this ratio is always bounded by some absolute (not depending on dimension) constant.

In this paper we show that this conjecture is not true.

The main result is the following theorem

Theorem 0.1. *There is a sequence of square ($N_n \times N_n$, $N_n \to \infty$) matrices A_n such that* $\lim_n \dfrac{\overline{\mu}(A_n)}{\mu(A_n)} = \infty$.

The paper is organized as follows: in Section 1 we prove an infinitedimensional analog (construct an example) of the main result. We are not going to discuss whether this analog is a correct one, because in Section 2 we prove the main theorem by discretizing the example from Section 1.

The technique used is quite new for control theory.

The example in Section 1 is based on standard facts in the theory of singular integral equations. I will present them without proofs, since they can be found in many textbooks.

The discretization is much more involved, although the experts in the field (of singular integral equations) would definitely recognize the ideas from localization technique for finite section methods. However, standard results would not work in our case, and one has to "push existing technique to a limit" to prove Theorem 0.1.

One can also consider different classes of diagonal uncertainties Δ, such us Linear Time Varying, or Nonlinear Time Invariant, or Nonlinear Time Varying, introduce corresponding structured norms and their upper bounds. This problem is already solved, see [5, 4, 8, 7]. The upper bound is the same for all three above mentioned classes of uncertainties and it coincides with corresponding structured norms.

This illustrates a well known principle in robust stability, that the more you know about uncertainty, the more difficult mathematical problem you have to solve to find the stability margin.

In conclusion I should mention some problems that are still open.

First of all the example constructed does not give any idea of how fast the ratio $\overline{\mu}/\mu$ can grow with the dimension. It was shown by A. Megretskii [3], that the ratio $\overline{\mu}/\mu$ can be estimated by $C \cdot n$, where C is an absolute constant.

I suspect that this estimate is very conservative, and that sharp estimate should grow much slower, like $C \log n$. That would explain why numerical experiment did not yield big ratio $\overline{\mu}/\mu$.

So, the problem is to find (more or less) sharp estimates on the ratio $\overline{\mu}/\mu$. Both, asymptotic and estimates for small n are interesting.

Another problem is to find an algebraic example, not requiring advanced "hard analysis". One of the possible ways here is to take a trivial discretization of the example in Section 1 (replace integrals by Riemann sums) and show directly that the resulting operators (matrices) have small μ. Note, that it is not known to me whether such operators indeed have small μ.

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1. An infinitedimensional analog of main result

Let us first solve the problem, which is a infinitedimesional analog of our. Let $L^2 = L^2(\nu)$, where ν is some measure. For an operator A on $L^2(\nu)$ define it structured singular value $\mu(A)$ as

$$\mu(A) = \Big(\inf\{\|\varphi\|_\infty \,:\, \varphi \in L^\infty(\mu),\ I - M_\varphi A \text{ is not invertible}\}\Big)^{-1};$$

here M_φ is the multiplication operator by φ, $M_\varphi f = \varphi f$. Sometimes, when it will not lead to a confusion, we will use the symbol φ the operator M_φ. For example, φA always denotes the operator $M_\varphi A$, but to use $A\varphi$ for AM_φ is ambiguous.

We can also define an upper bound $\overline{\mu}(A)$ by

$$\overline{\mu}(A) := \inf\{\|M_\psi A M_\psi^{-1}\| : \psi, \psi^{-1} \in L^\infty(\mu)\};$$

here again M_ψ denotes the multiplication operators by ψ.

A simple observation: if ν is a finite combination of atoms, then the operator A is an operator in a finitedimensional space (matrix), and the definitions of μ and $\overline{\mu}$ coincide with ones given above in Section 0

We will construct a measure ν on the complex plane $\mathbb{C}$ and an operator A on $L^2(\mu)$ such that the ratio $\dfrac{\overline{\mu}(A)}{\mu(A)}$ is as large as we want.

Let Γ be a simple $\mathcal{C}^2$-smooth closed curve on $\mathbb{C}$. Our measure ν is just the arclength on Γ. Consider the singular integral operator (Cauchy integral) $S = S_\Gamma$ defined by

$$(Sf)(t) = \frac{1}{\pi i}\,\text{p.v.}\int_\Gamma \frac{f(s)}{s-t}ds$$

Clearly, the operator is well defined on smooth functions. It is also well known that for any $f \in L^2$ the principal value Sf exists a. e., and that the operator S is bounded on $L^2(\mu)$, see for example [2].

The structure of the operator S is well known. Namely, let

$$P = \frac{1}{2}(I+S), \qquad Q = \frac{1}{2}(I-S).$$

Then $S = P - Q$ and P, Q are complimentary projections ($P + Q = I$, $P^2 = P$, $Q^2 = Q$), not necessarily orthogonal. The projections P and Q are projections onto subspaces $L^2_\pm = L^2(\Gamma)_\pm$, $L^2_\pm := \operatorname{clos}_{L^2} \operatorname{Rat}_\pm$; here Rat_+ denotes all rational functions with poles outside of Γ, and Rat_- consists of all rational functions f, $f(\infty) = 0$ with poles inside Γ.

If Γ is the unit circle, the spaces $L^2_\pm$ coincide with classical Hardy spaces $H^2_\pm$, and projections P and Q are orthogonal.

Operator S can be defined for non-compact contours Γ as well. An important case we will need is when Γ is just the real line $\mathbb{R}$. In this case the projections P and Q are orthogonal projections onto the Hardy spaces in the upper and lower half planes respectively. Since the projections P and Q are orthogonal, $\|S_\mathbb{R}\| = 1$. We will use this fact in what follows.

Theorem 1.1. *Let Γ be a simple $\mathcal{C}^2$-smooth closed curve in $\mathbb{C}$. Let $\varphi \in L^\infty(\Gamma)$, $\|\varphi\|_\infty < 1$. Then the operator $I - \varphi S_\Gamma$ is invertible.*

Proof. The theorem is just a restatement of a well known fact about singular integral equations. Using the fact that $S = P - Q$, $I = P + Q$ we can write

$$I - \varphi S = P + Q - \varphi\,(P+Q) = (1-\varphi)P + (1+\varphi)Q = (1+\varphi)\Big(\frac{1-\varphi}{1+\varphi}P + Q\Big).$$

Since $\|\varphi\|_\infty < 1$, the operator of multiplication by $1+\varphi$ is invertible. Therefore $I-\varphi S$ is invertible if and only if the operator $aP+Q$ is invertible, where $a = \dfrac{1-\varphi}{1+\varphi}$.

Note, that since $\|\varphi\|_\infty < 1$, we have $a, a^{-1} \in L^\infty$, and moreover, the range of a lies in a sector with vertex at the origin and the opening angle strictly less than π.

A well known theorem about singular integral operators, see for example [2, Theorem 3.1 in Chapter 12] says that even under weaker assumptions on the curve Γ, the operator $aP+Q$ is invertible if $a, a^{-1} \in L^\infty$, and the range of a lies in a sector with vertex at the origin and the opening angle strictly less than π. ☐

Let us now show that $\inf\{\|\psi S\psi^{-1}\| : \psi, \psi^{-1} \in L^\infty\}$ can be as large as we want. For a curve Γ let us define its *Ahlfors constant* $A(\Gamma)$ by

$$A(\Gamma) = \sup \frac{|B_r(x) \cap \Gamma|}{r};$$

here the *supremum* is taken over all discs $B_r(x) = \{z \in \mathbb{C} : |z-x| < r\}$, and $|X|$ denotes the length of the set X.

Theorem 1.2. *For any $\psi \in L^\infty(\Gamma)$ with $\psi^{-1} \in L^\infty(\Gamma)$ we have*

$$\|M_\psi S_\Gamma M_\psi^{-1}\| \geqslant c \cdot A(\Gamma),$$

where $c > 0$ is an absolute constant.

This theorem immediately implies that one can construct a $\mathcal{C}^2$-smooth contour Γ such that $\dfrac{\overline{\mu}(S_\Gamma)}{\mu(S_\Gamma)}$ is as large as we want. Indeed, Theorem 1.1 means that $\mu(S_\Gamma) \leqslant 1$. On the other hand, Theorem 1.2 implies that $\overline{\mu}(\Gamma) \geqslant cA(\Gamma)$, where c is the absolute constant from the theorem. And it is very easy to construct a $\mathcal{C}^2$-smooth contour Γ with arbitrary large Ahlfors constant $A(\Gamma)$, see Fig. 3.

Proof of Theorem 1.2. First of all pick two orthogonal directions such that the contour Γ does not contain a straight segment parallel to one of them. It is always possible, because a smooth contour can contain at most countably many straight segments.

Let $B_r(x)$ be a disc such that $|\Gamma \cap B_r(x)| \geqslant 0.8A(\Gamma)r$. Using straight lines parallel to the chosen directions we can split the disc $B_r(x)$ into four parts B_k, $k = 1, ..., 4$, such that $|\Gamma \cap B_k| = |\Gamma \cap B_r(x)|/4$, see Fig. 4. (On this figure we first used a horizontal line to split $\Gamma \cap B_r(x)$ into halves of equal length, and then used the vertical lines to split each half.)

Among the parts B_k pick two that touch at most at a point (shaded regions on Fig. 4). Without loss of generality we can call them B_1 and B_2. Clearly the set $\{z_1 - z_2 : z_{1,2} \in B_{1,2}\}$ is contained in a sector with vertex at the origin and with an opening angle at most $\pi/2$. Therefore there exist

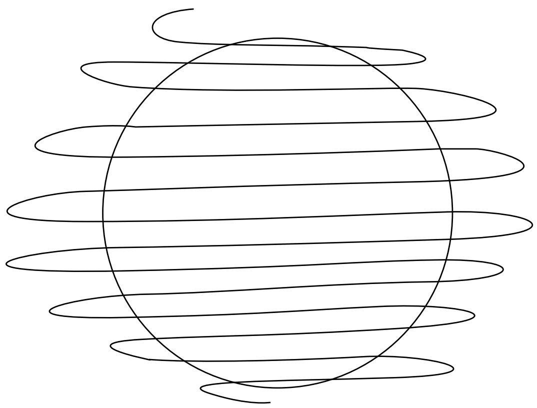

FIGURE 3. A curve Γ with large Ahlfors constant $A(\Gamma)$

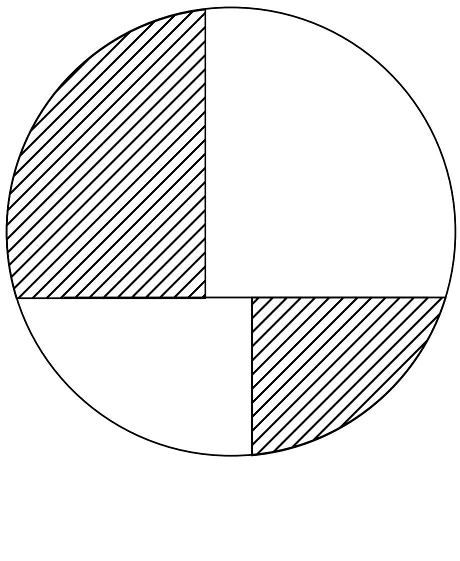

FIGURE 4. Each of four parts of the disk contains equal length of Γ

$\xi \in \mathbb{C}$, $|\xi| = 1$ such that

$$\operatorname{Re}\Big(\xi \cdot \frac{1}{z_1 - z_2}\Big) \geqslant \frac{1}{\sqrt{2}} \cdot \frac{1}{|z_1 - z_2|} \geqslant \frac{1}{2\sqrt{2}r}, \qquad \forall z_{1,2} \in B_{1,2}\,. \tag{1.1}$$

Let $\Gamma_{1,2} := \Gamma \cap B_{1,2}$. Pick $\psi \in L^\infty$ such that $\psi^{-1} \in L^\infty$. Without loss of generality we can assume that

$$\int\limits_{\Gamma_1} |\psi|^2 |dz| \leqslant \int\limits_{\Gamma_2} |\psi|^2 |dz|$$

To estimate the norm of the operator $M_\psi S M_\psi^{-1}$, pick two test functions

$$f := (\overline{\psi})^{-1} \chi_{\Gamma_1} \frac{|dz|}{dz} \qquad g := \psi \chi_{\Gamma_2}\,;$$

here $\dfrac{|dz|}{dz}$ is the recipical of the direction $\dfrac{dz}{|dz|}$ of the tangent line to Γ at the point z. Note, that $\big||dz|/dz\big| = 1$. Note also that

$$\|f\|_2^2 = \int\limits_{\Gamma_1} |\psi|^{-2} |dz|, \qquad \|g\|_2^2 = \int\limits_{\Gamma_2} |\psi|^2 |dz|.$$

We can estimate using (1.1)

$$\begin{aligned}
\big|(\psi S \psi^{-1} f, g)\big| &= \frac{1}{\pi} \Big| \int\limits_{\Gamma_2} \Big(\int\limits_{\Gamma 1} \frac{f(z_1)}{z_1 - z_2}\, dz_1 \Big) \overline{g}(z_2)\, |dz_2| \,\Big| \\
&= \frac{1}{\pi} \Big| \iint\limits_{\Gamma_1 \times \Gamma_2} \xi \cdot \frac{|\psi(z_1)|^{-2} |\psi(z_2)|^2}{z_1 - z_2}\, |dz_1||dz_2| \,\Big| \\
&\geqslant \frac{1}{\pi} \cdot \frac{1}{2\sqrt{2} \cdot r} \int_{\Gamma_1} |\psi|^{-2} |dz_1| \cdot \int_{\Gamma_2} |\psi|^2\, |dz_2| \\
&= \frac{1}{\pi} \cdot \frac{1}{2\sqrt{2} \cdot r} \Big(\int_{\Gamma_1} |\psi|^{-2} |dz_1| \cdot \int_{\Gamma_2} |\psi|^2\, |dz_2| \Big)^{1/2} \|f\|_2 \|g\|_2
\end{aligned}$$

Cauchy–Schwartz inequality implies

$$\begin{aligned}
|\Gamma_1|^2 = \Big(\int_{\Gamma_1} 1\, |dz| \Big)^2 &\leqslant \int_{\Gamma_1} |\psi|^{-2} |dz| \cdot \int_{\Gamma_1} |\psi|^2\, |dz| \\
&\leqslant \int_{\Gamma_1} |\psi|^{-2} |dz| \cdot \int_{\Gamma_2} |\psi|^2\, |dz|
\end{aligned}$$

(in the last inequality we used the assumption that $\int_{\Gamma_1} |\psi|^2 |dz| \leqslant \int_{\Gamma_2} |\psi|^2 |dz|$).

So we get

$$\big|(\psi S \psi^{-1} f, g)\big| \geqslant \frac{|\Gamma_1|}{2\sqrt{2}\pi r} \|f\|_2 \|g\|_2\,.$$

By the construction

$$|\Gamma_1| = |\Gamma \cap B_r(x)| \geqslant \frac{1}{4} r \cdot 0.8 A(\Gamma)$$

so $\|M_\psi S M_\psi^{-1}\| \geqslant \dfrac{1}{10\sqrt{2}\pi} A(\Gamma)$. □

2. The finitedimensional case

In this section we are going to show how to construct a finitedimensional operator (matrix) A with arbitrary large ratio $\overline{\mu}(A)/\mu(A)$. The operator we construct is just a discretization of the operator $S = S_\Gamma$ from the previous section.

For the curve Γ from the previous section consider the sequence of its partitions $\mathcal{P}_n$ into finitely many arcs. For the simplicity we assume that each partition $\mathcal{P}_n$ is a refinement of the previous $\mathcal{P}_{n_1}$. We also assume that the maximal length of an arc in the partition $\mathcal{P}_n$ tends to 0 as $n \to \infty$.

Let P_n be the averaging operators,

$$P_n f = \sum_{k=1}^{N(n)} \Big(\frac{1}{|\Delta_k^n|} \int\limits_{\Delta_k^n} f(z)\, |dz| \Big) \chi_{\Delta_k^n} ,$$

where Δ_k^n, $k = 1, 2, ..., N(n)$ are the arcs of the partition $\mathcal{P}_n$.

Clearly, P_n are orthogonal projections on $L^2(\Gamma)$. Let $X_n := \operatorname{Range} P_n = P_n L^2$. Define operators S_n by

$$S_n = P_n S \bigm| X_n.$$

Operators S_n are operators on finitedimensional spaces (matrices). Consider a natural orthonormal basis $\{|\Delta_k^n|^{-1/2}\chi_{\Delta_k^n} : k = 1, 2, ..., N(n)\}$ in X_n. We can treat the operator S_n as matrix in this natural basis. Diagonal operators in this basis are just multiplications by functions $\varphi \in X_n$.

Modifying a little the reasoning in Theorem 1.2, we can show that

$$\lim_n \Big(\inf\{\|D S_n D^{-1}\| : D \text{ is diagonal}\} \Big) \geqslant c A(\Gamma). \tag{2.1}$$

The main technical difficulty here is that the curve Γ is now "quantized", so to get the estimate we need only count arcs Δ_k^n completely contained in regions B_1, B_2. But this difficulty is easy to overcome: as $n \to \infty$ the total length of arcs Δ_k^n that is intersected by the lines tends to 0. We leave all technical details here to the reader.

We are going to show that $\lim_n \mu(S_n) \leqslant 10$. Together with (2.1) this implies that the ratio $\overline{\mu}(S_n)/\mu(S_n)$ can be made as large as we want (by picking an appropriate contour Γ and sufficiently large n).

We will show that (given a sequence of partitions $\mathcal{P}_n$) there exists $N > 0$ such that for all $n \geqslant n$ and all $\varphi_n \in X_n$, $\|\varphi_n\|_\infty \leqslant 0.1$ the operators $I_n - \varphi_n S_n$ are invertible.

Suppose, that is not true. Then there exists a subsequence $n_k \to \infty$ such that for any n in this subsequence there exists a vector $f_n \in X_n$ and a function $\varphi_n \in X_n$, $\|\varphi_n\|_\infty \leqslant 0.1$, such that $f_n = \varphi_n S_n f_n$, and therefore

$$|f_n(t)| \leqslant 0.1 \cdot |(S_n f_n)(t)| \qquad \forall t \in \Gamma. \tag{2.2}$$

By taking a subsequence, we can always assume that (2.2) holds for all $n > N$. Let us show that this is impossible.

We need few technical lemmas.

For a set Δ let $E(\Delta)$ denote the operator of multiplication by χ_Δ, $E(\Delta)f := \chi_\Delta f$.

Lemma 2.1 (Local norm lemma). *For any point $\tau \in \Gamma$ and any sequence of arcs $\Delta_n \ni \tau$ such that $|\Delta_n| \to 0$ we have*

$$\limsup_{n\to\infty} \|E(\Delta_n)SE(\Delta_n)\| \leqslant 1$$

Note, that it is possible to prove that $\lim_n \|E(\Delta_n)SE(\Delta_n)\| = 1$, but we will only use the statement of the lemma.

Proof. The proof is based on the fact that the operator behave S is locally almost as the operator $S_{\mathbb{R}}$, which has norm 1.

To write a formal proof, consider an arc $\Delta \subset \Gamma$ containing all Δ_n. Let $\varphi : \Delta' \to \Delta$ be the arc-length parametrization of the arc Δ. Let $\Delta'_n := \varphi^{-1}(\Delta_n)$. Clearly, $|\Delta'_n| = |\Delta_n| \to 0$.

Define the unitary operator $U : L^2(\Delta') \to L^2(\Delta)$, where $\Delta' = \varphi^{-1}(\Delta)$ by $Uf = f \circ \varphi$. The restriction of U onto $L^2(\Delta'_n)$ is a unitary operator from $L^2(\Delta'_n)$ onto $L^2(\Delta'_n)$.

Let $\widetilde{S} := U^{-1}E(\Delta)SE(\Delta)U$ One can write

$$[\widetilde{S}E(\Delta'_n)f](t) = \frac{1}{\pi i}\int\limits_{\Delta'_n} \frac{\varphi'(s)}{\varphi(s)-\varphi(t)} f(\varphi(s))\,ds, \qquad t \in \Delta'.$$

One can estimate

$$\left|\frac{1}{s-t} - \frac{\varphi'(s)}{\varphi(s)-\varphi(t)}\right| = \left|\frac{\varphi(s)-\varphi(t)-\varphi'(s)(s-t)}{(s-t)(\varphi(s)-\varphi(t))}\right| \leqslant \text{Const}$$

since $\varphi(s)$ is $\mathcal{C}^2$-smooth. This implies that

$$\|E(\Delta'_n)\widetilde{S}E(\Delta'_n) - E(\Delta'_n)S_{\mathbb{R}}E(\Delta'_n)\| \to 0.$$

But we know that $\|S_{\mathbb{R}}\| = 1$, and the lemma is proved. □

Lemma 2.2 (Localization Principle). *Let $S = S_\Gamma$, where Γ is a closed rectifiable curve (not necessarily smooth). Suppose that $\Delta_1, \Delta_2 \subset \Gamma$, $\operatorname{dist}(\Delta_1, \Delta_2) > 0$. Then the operator $E(\Delta_1)SE(\Delta_2)$ is compact.*

Proof. Trivial. □

Let us remind that the essential norm $\|A\|_{\text{ess}}$ of an operator A is the distance from the operator A to compact operators.

Lemma 2.3 (Essential Norm Lemma).

$$\|S_{\Gamma}\|_{\text{ess}} \leqslant 2$$

Proof. Pick $\varepsilon > 0$. Applying Lemma 2.1 find for each point $\tau \in \Gamma$ an open arc Δ, $\tau \subset \Delta$ such that $\|E(\Delta)SE(\Delta)\| \leqslant 1 + \varepsilon$. Pick a finite covering Δ_k, $k = 1, 2, ..., N$ of Γ by such arcs. Since Γ is a one-dimensional set, we can always chose a covering such that each point of Γ is covered by at most 2 arcs Δ_k.

Split Γ into finitely many disjoint arcs Δ'_k, $k = 1, 2, ..., N$ such that each arc Δ'_k is strictly inside Δ_k (i. e. such that $\operatorname{dist}(\Delta'_k, \Gamma \setminus \Delta_k) > 0$). Then

$$S = \sum_k SE(\Delta'_k) = \sum_k E(\Delta_k)SE(\Delta'_k) \sum_k E(\Gamma \setminus \Delta_k)SE(\Delta'_k).$$

The norm of the first sum is estimated by $2(1 + \varepsilon)$, the second sum is a compact operator. Therefore $\|S\|_{\text{ess}} \leqslant 2 + 2\varepsilon$. Since the inequality holds for any $\varepsilon > 0$, the lemma is proved. $\square$

Lemma 2.4. *Let a sequence f_n of vectors in a Hilbert space converge weakly to a vector f. Then*

$$\limsup_{n\to\infty} \|f_n\|^2 = \|f\|^2 + \limsup_{n\to\infty} \|f - f_n\|^2.$$

Proof. Let P be the orthogonal projection onto the linear span of f. The condition $f_n \xrightarrow{w} f$ implies $\|f - Pf_n\| \to 0$. The lemma follows immediately. $\square$

Suppose now that we found a sequence of vectors $f_n \in X_n$, normalized by $\|f_n\|_2 = 1$ and such, that

$$|f_n(t)| \leqslant 0.1 \cdot |(S_n f_n)(t)| \qquad \forall t \in \Gamma. \tag{2.3}$$

Taking a subsequence, if necessary, we can assume weak convergence $f_n \xrightarrow{w} f$.

Consider the simple case $f = 0$ first. The condition $|f_n(t)| \leqslant 0.1 \cdot |(S_n f_n)(t)|$ implies that $\|f_n\|_2 \leqslant 0.1\|P_n S f_n\|_2 \leqslant 0.1\|Sf_n\|_2$.

On the other hand, by Lemma 2.3, there exists a compact operator K such that $\|S - K\| \leqslant 3$. Therefore

$$1 = \limsup_{n\to\infty} \|f_n\|_2 \leqslant 0.1 \limsup_{n\to\infty} \|Sf_n\|_2 = 0.1 \limsup_{n\to\infty} \|(S - K)f_n\|_2 \leqslant 0.3,$$

and we get a contradiction!

Let us now suppose that $f \neq 0$.

We will say that an arc $\Delta \subset \Gamma$ *agree* with the sequence of partitions $\mathcal{P}_n$ if Δ is a union of arcs Δ^n_k from some partition $\mathcal{P}_n$. If an arc Δ agree with the sequence of partitions $\mathcal{P}_n$, the inequality (2.3) implies

$$\begin{aligned}
10 \limsup_{n\to\infty} \|E(\Delta)f_n\|_2 &\leqslant \limsup_{n\to\infty} \|E(\Delta)P_n S f_n\|_2 \\
&\leqslant \limsup_{n\to\infty} \|E(\Delta)Sf_n\|_2 \\
&\leqslant \limsup_{n\to\infty} \|E(\Delta)Sf\|_2 + \limsup_{n\to\infty} \|E(\Delta)S(f - f_n)\|_2
\end{aligned}$$

Let $g_n := f - f_n$. Clearly $f_n \to f$ weakly, and $\limsup_n \|g_n\|_2 \leqslant 1$.

In the following lemma the sequence $g_n \overset{w}{\to} 0$ is supposed to be fixed.

Lemma 2.5. *For any point $\tau \in \Gamma$ and any positive ε, δ, there exists an arc $\Delta \ni \tau$, $|\Delta| \leqslant \varepsilon$, which agree with $\mathcal{P}_n$ and such that*

$$\limsup_{n\to\infty} \|E(\Delta)Sg_n\|_2 \leqslant 2 \limsup_{n\to\infty} \|E(\Delta)g_n\|_2 + |\Delta| \cdot \delta.$$

Proof. For a measurable set Δ define $\rho(\Delta) := \limsup_n \|E(\Delta)g_n\|_2$. Clearly, $\rho(\cdot)$ is a subadditive function, $\rho(\Delta_1 \cup \Delta_2) \leqslant \rho(\Delta_1) + \rho(\Delta_2)$. By Lemma 2.1, for any sufficiently small Δ' ($|\Delta'| < \alpha(\tau)$) we have

$$\limsup_n \|E(\Delta')SE(\Delta')\| \leqslant \sqrt{2}.$$

On the other hand, Lemma 2.2 implies that for any arc Δ strictly inside Δ' ($\operatorname{dist}(\Delta, \Gamma \setminus \Delta') > 0$) the operator $E(\Delta)SE(\Gamma \setminus \Delta')$ is compact.

Combining the above two facts and using $g_n \overset{w}{\to} 0$ we get that for all sufficiently small Δ

$$\begin{aligned}
\limsup_{n\to\infty} \|E(\Delta)Sg_n\|_2 &= \limsup_{n\to\infty} \|E(\Delta)SE(\Delta')g_n\|_2 \\
&\leqslant \limsup_{n\to\infty} \|E(\Delta')SE(\Delta')\| \cdot \|E(\Delta')g_n\|_2 \\
&\leqslant \sqrt{2} \limsup_{n\to\infty} \|E(\Delta')g_n\|_2 = \sqrt{2}\rho(\Delta')
\end{aligned}$$

If there exist small arcs Δ, Δ', Δ is strictly inside Δ' such that $\rho(\Delta) \leqslant \sqrt{2}\rho(\Delta')$ we get

$$\limsup_{n\to\infty} \|E(\Delta)Sg_n\| \leqslant \sqrt{2}\rho(\Delta') \leqslant 2\rho(\Delta) = 2 \limsup_n \|E(\Delta)g_n\|_2,$$

and we are done.

Suppose there are no such Δ and Δ'. Then there exists a strictly decreasing (Δ_{k+1} is strictly inside of Δ_k) sequence of arcs Δ_k, $k = 1, 2, \dots$ containing x, arcs Δ_k agree with $\mathcal{P}_n$, such that

$$\frac{|\Delta_k|}{|\Delta_{k+1}|} \leqslant 1.1, \qquad \text{but} \qquad \frac{\rho(\Delta_k)}{\rho(\Delta_{k+1})} > \sqrt{2}.$$

We know that $\rho(\Delta_1) \leqslant \rho(\Gamma) \leqslant 1$. Therefore the above inequalities imply that for sufficiently large k we have for

$$\rho(\Delta_k) \leqslant \frac{\delta}{\sqrt{2}} \cdot |\Delta_{k+1}|.$$

This implies for $\Delta = \Delta_{k+1}$, $\Delta' = \Delta_k$

$$\limsup_{n\to\infty} \|E(\Delta)Sg_n\| \leqslant \sqrt{2}\rho(\Delta') \leqslant \delta|\Delta|.$$

$\square$

Now we are in a position to complete the proof of Theorem 0.1. By Lemma 2.4

$$\limsup_{n\to\infty} \|E(\Delta)f_n\| = \sqrt{\|E(\Delta)f\|^2 + \limsup_{n\to\infty}\|E(\Delta)g_n\|^2}$$
$$\geqslant \frac{1}{\sqrt{2}}\Big(\|E(\Delta)f\| + \limsup_{n\to\infty}\|E(\Delta)g_n\|\Big),$$

where $g_n = f - f_n$. We know that

$$\|E(\Delta)f_n\| \leqslant \frac{1}{10}\cdot\|E(\Delta)Sf_n\| \leqslant \frac{1}{10}\big(\|E(\Delta)Sf\| + \|E(\Delta)Sg_n\|\big).$$

Therefore by Lemma 2.5 for any point $\tau \in \Gamma$ and any positive ε, (put $\delta = 1$) one can find an arc $\Delta \ni \tau$, $|\Delta| < \varepsilon$, which agree with $\mathcal{P}_n$ and

$$\begin{aligned}\|E(\Delta)f\| + \limsup_{n\to\infty}&\|E(\Delta)g_n\|\\ &\leqslant \frac{\sqrt{2}}{10}\Big(\|E(\Delta)Sf\| + \limsup_{n\to\infty}\|E(\Delta)Sg_n\|\Big)\\ &\leqslant \frac{\sqrt{2}}{10}\Big(\|E(\Delta)Sf\| + 2\limsup_{n\to\infty}\|E(\Delta)g_n\| + |\Delta|\Big).\end{aligned}$$

Hence

$$\|E(\Delta)f\| \leqslant \frac{\sqrt{2}}{10}(\|E(\Delta)Sf\| + |\Delta|).$$

This implies

$$\|E(\Delta)f\|^2 \leqslant \frac{1}{50}\big(\|E(\Delta)Sf\|^2 + 2|\Delta|\cdot\|E(\Delta)Sf\| + |\Delta|^2\big)$$

Pick a sequence of such intervals $\Delta \ni \tau$, $|\Delta| \to 0$. Dividing the above inequality by $|\Delta|$ and taking $\limsup\limits_{|\Delta|\to 0}$ we get

$$\limsup_{|\Delta|\to 0}\frac{1}{|\Delta|}\int_\Delta |f(z)|^2\,|dz| \leqslant \frac{1}{50}\limsup_{|\Delta|\to 0}\frac{1}{|\Delta|}\int_\Delta |\big(Sf\big)(z)|^2\,|dz|$$

According to Lebesgue density theorem, cf [10, Theorem 7.16] for any locally L^1 function F limit

$$\lim_{|\Delta|\to 0, x\in\Delta}\frac{1}{|\Delta|}\int_\Delta F = F(x)$$

for almost all x; here the limit is taken over intervals (arcs) containing x.

Therefore

$$|f(\tau)| \leqslant \frac{\sqrt{2}}{10}|\big(Sf\big)(\tau)|$$

for almost all τ. Hence, there exists a function $\varphi \in L^\infty$, $\|\varphi\|_\infty \leqslant \frac{\sqrt{2}}{10}$, such that

$$f = \varphi Sf.$$

But, by Theorem 1.1, the operator $I - \varphi S$ is invertible, so we got a contradiction.

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