A Simple Proof of Hunt – Muckenhoupt – Wheeden Theorem.

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First draft.

1. A weighted imbedding theorem.

Let $v(\lambda)$ denotes the harmonic extension of a function v defined on a unit circle $\mathbb{T}$ at the point $\lambda \in \mathbb{D}$.

Theorem 1.1 *Let a weight $v \in L^1(\mathbb{T})$ and a measure μ in $\mathbb{D}$ satisfy*

$$\int_{\mathbb{D}} \frac{1-|\lambda|^2}{|1-\overline{\lambda}z|^2} v(z)^2 d\mu(z) \leq Cv(\lambda)$$

where the constant C does not depend on $\lambda \in \mathbb{D}$. Then the following imbedding theorem

$$\int_{\mathbb{D}} \left|(fv^{1/2})(z)\right|^2 d\mu(z) \leq K(C) \int_{\mathbb{T}} |f|^2, \qquad \forall f \in L^2(T) \tag{1.1}$$

holds.

Corollary 1.2. *Let w be a weight on $\mathbb{T}$ such that $w^{-1} \in L^1(\mathbb{T})$, and μ be a measure in $\mathbb{D}$. If*

$$\int_{\mathbb{D}} \frac{1-|\lambda|^2}{|1-\overline{\lambda}z|^2} [w^{-1}(z)]^2 d\mu(z) \leq Cw^{-1}(\lambda)$$

then the following weighted imbedding theorem

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \leq K(C) \int_{\mathbb{T}} |f|^2 w, \qquad \forall f \in L^2(T, v)\,. \tag{1.2}$$

holds.

Remark. It is easy to show that the condition $w^{-1} \in L^1$ is necessary for the existence of a nonzero measure μ satisfying (1.2).

This Corollary follows immediately from Theorem 1 if we put $v = w^{-1}$ and note that the imbedding (1.2) is equivalent to (1.1).

To prove Theorem 1 we need the following well known test of boundedness of an integral operator, see [1]

Theorem 1.3 (Senichkin – Vinogradov Test). *Let $\mathcal{Z}$ be a measure space, k be a locally summable nonnegative measurable function on $\mathcal{Z} \times \mathcal{Z}$. Let*

$$\int_{\mathcal{Z}} k(s,t)k(s,x)ds \le C\left[k(x,t) + k(t,x)\right], \qquad \text{a.e. on } \mathcal{Z} \times \mathcal{Z}.$$

Then for any positive $g \in L^2(\mathcal{Z})$

$$\iint_{\mathcal{Z}\times\mathcal{Z}} k(s,t)g(s)g(t)dsdt \le 2C\|g\|^2_{L^2(\mathcal{Z})},$$

and the integral operator with kernel $k(s,t)$ is bounded in $L^2(\mathcal{Z})$.

We present a very simple proof of the theorem only for the sake of completeness.
Proof of Theorem 1. Let $f \in L^2$, $\|f\|_2 = 1$ be a nonnegative functions such that

$$Q \stackrel{\text{def}}{=} \iint_{\mathcal{Z}\times\mathcal{Z}} k(s,t)f(s)f(t)dsdt < \infty.$$

Then

$$\begin{aligned} Q^2 = \left(\int_{\mathcal{Z}} f(s)\left(\int_{\mathcal{Z}} k(s,t)f(t)dt\right)ds\right)^2 \le \|f\|^2_{L^2(\mathcal{Z})} \cdot \left\|\int_{\mathcal{Z}} k(.,t)f(t)dt\right\|^2 \le \\ \le \int_{\mathcal{Z}} ds \left(\iint_{\mathcal{Z}\times\mathcal{Z}} k(s,t)k(s,x)f(t)f(x)dtdx\right) = \\ = \iint_{\mathcal{Z}\times\mathcal{Z}} f(t)f(x)\left(\int_{\mathcal{Z}} k(s,t)k(s,x)ds\right) \le \\ \le 2C\int_{\mathcal{Z}} [k(t,x) + k(t,x)]f(t)f(x)dtdx = 2CQ. \end{aligned}$$

Therefore the corresponding integral operator is bounded and its norm can be estimated above as $\sqrt{2C}$. □

Proof of Theorem 1. Let $\mathcal{J}$ be the imbedding operator that we want to prove is bounded, $\mathcal{J} : L^2 \to L^2(\mu)$,

$$\mathcal{J}f(z) = (fv^{1/2})(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\overline{z}t|^2} f(t)v^{1/2}(t)dm(t).$$

Then the (formal) conjugate to $\mathcal{J}$ operator $\mathcal{J}^*$, $\mathcal{J}^* : L^2(\mu) \to L^2$ can be written as

$$\mathcal{J}^*f(t) = v^{1/2}(t)\int_{\mathbb{D}} \frac{1-|z|^2}{|1-\overline{z}t|^2} f(z)d\mu(z),$$

so

$$\|\mathcal{J}^* f\|^2 = \int_{\mathbb{D}}\int_{\mathbb{D}}\left(\int_{\mathbb{T}} v(t)\frac{1-|z|^2}{|1-\overline{z}t|^2}\frac{1-|s|^2}{|1-\overline{s}t|^2}dm(t)\right) f(z)\overline{f(s)}d\mu(z)d\mu(s) \tag{1.3}$$

It is easy to estimate the inner integral. Namely,

$$|1-\overline{s}z| = |1-\overline{s}t+\overline{s}t-\overline{s}z| \le |1-\overline{s}t| + |s|\cdot|t-z| \le |1-\overline{s}t| + |1-\overline{z}t|,$$

so

$$|1-\overline{s}t|^{-1}|1-\overline{z}t|^{-1} \le |1-\overline{s}z|^{-1}\left(|1-\overline{s}t|^{-1}+|1-\overline{z}t|^{-1}\right),$$

and finally

$$|1-\overline{s}t|^{-2}|1-\overline{z}t|^{-2} \le 2\cdot|1-\overline{s}z|^{-2}\left(|1-\overline{s}t|^{-2}+|1-\overline{z}t|^{-2}\right). \tag{1.4}$$

Hence we can estimate the inner integral in (1.3):

$$\int_{\mathbb{T}} v(t)\frac{1-|z|^2}{|1-\overline{z}t|^2}\frac{1-|s|^2}{|1-\overline{s}t|^2}dm(t) \le$$
$$\frac{2}{|1-\overline{s}z|^2}\left\{\int_{\mathbb{T}} v(t)\frac{(1-|z|^2)(1-|s|^2)}{|1-\overline{s}t|^2}dm(t) + \int_{\mathbb{T}} v(t)\frac{(1-|s|^2)(1-|z|^2)}{|1-\overline{z}t|^2}dm(t)\right\} \le$$
$$\le 2\left\{\frac{(1-|z|^2)}{|1-\overline{s}z|^2}v(s) + \frac{(1-|s|^2)}{|1-\overline{s}z|^2}v(z)\right\}$$

Therefore

$$\|\mathcal{J}^* f\|^2 \le 4\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{1-|s|^2}{|1-\overline{s}z|^2}v(z)|f(z)|\cdot|f(s)|d\mu(z)d\mu(s)$$

Now we apply Senichkin – Vinogradov test (Theorem 1) to the integral operator in $L^2(\mu)$ with kernel K

$$K(z,s) = \frac{1-|s|^2}{|1-\overline{s}z|^2}v(z)$$

Clearly

$$\int_{\mathbb{D}} K(z,s)K(z,x)d\mu(z) = \int_{\mathbb{D}}\frac{1-|s|^2}{|1-\overline{s}z|^2}\cdot\frac{1-|x|^2}{|1-\overline{x}z|^2}\left[v(z)\right]^2 d\mu(z) \le$$
$$\le \frac{2}{|1-\overline{s}x|^2}\left\{\int_{\mathbb{D}}\frac{(1-|s|^2)(1-|x|^2)}{|1-\overline{s}z|^2}\left[v(z)\right]^2 d\mu(z) + \right.$$
$$\left. + \int_{\mathbb{D}}\frac{(1-|s|^2)(1-|x|^2)}{|1-\overline{x}z|^2}\left[v(z)\right]^2 d\mu(z)\right\} \le$$
$$\le 2C\left\{\frac{1-|x|^2}{|1-\overline{s}x|^2}v(s) + \frac{1-|s|^2}{|1-\overline{s}x|^2}v(x)\right\} \le 2C(K(s,x)+K(x,s))$$

$\square$

2. Carleson measure.

For a function f on the circle $\mathbb{T}$ let $f(z)$, $z \in \mathbb{D}$ always denote the harmonic extension of f at z.

Theorem 2.1. *If a weight w satisfies the Muckenhoupt condition (A_2), then the measure $\frac{|\nabla w(z)|^2}{w(z)^2}(1-|z|^2)dxdy$ is Carleson.*

Proof. It is well known that the condition (A_2) can be rewritten as

$$\sup_{z\in\mathbb{D}}(w(z))(w^{-1}(z)) < \infty\,.$$

We can rewrite this as

$$\sup_{z\in\mathbb{D}}\left\{\left[w(z)/\exp\left\{(\log w)(z)\right\}\right]\cdot\left[(w^{-1})(z)/\exp\left\{(\log w^{-1})(z)\right\}\right]\right\} < \infty$$

By Jensen inequality expressions in brackets are at least 1, so

$$\sup_{z\in\mathbb{D}}\left[w(z)/\exp\left\{(\log w)(z)\right\}\right] \;= C < \infty\,,$$

or equivalently

$$\log\left\{w(z)\right\} - (\log w)\,(z) \le C \tag{2.1}$$

By Green's formula

$$\log\left\{w(s)\right\} - (\log w)\,(s) = -\frac{1}{2\pi}\iint_{\mathbb{D}}\log\left|\frac{1-\overline{s}z}{z-s}\right|\Delta\log\left\{w(z)\right\}\,dxdy =$$
$$\frac{1}{2\pi}\iint_{\mathbb{D}}\frac{|\nabla w(z)|^2}{w(z)^2}\log\left|\frac{1-\overline{s}z}{z-s}\right|\,dxdy = \frac{1}{4\pi}\iint_{\mathbb{D}}\frac{|\nabla w(z)|^2}{w(z)^2}\log\left|\frac{1-\overline{s}z}{z-s}\right|^2\,dxdy$$

Using an elementary inequality $\log(1/a) \ge 1-a$ for $0 < a \le 1$ we can estimate the last integral as

$$\iint_{\mathbb{D}}\frac{|\nabla w(z)|^2}{w(z)^2}\log\left|\frac{1-\overline{s}z}{z-s}\right|^2\,dxdy \ge \iint_{\mathbb{D}}\frac{|\nabla w(z)|^2}{w(z)^2}\left(1-\left|\frac{1-\overline{s}z}{z-s}\right|^2\right)\,dxdy =$$
$$= \iint_{\mathbb{D}}\frac{|\nabla w(z)|^2}{w(z)^2}\cdot\frac{(1-|s|^2)(1-|z|^2)}{|1-\overline{s}z|^2}\,dxdy$$

Together with (2.1) this imply

$$\iint_{\mathbb{D}}\frac{(1-|s|^2)}{|1-\overline{s}z|^2}\cdot\frac{|\nabla w(z)|^2}{w(z)^2}(1-|z|^2)\,dxdy \le C\,, \qquad \forall s\in\mathbb{D}\,,$$

that yields that the measure $\frac{|\nabla w(z)|^2}{w(z)^2}(1-|z|^2)\,dxdy$ is Carleson. $\square$

3. Equivalence of norms.

Let as usual given a function f on the unit circle $\mathbb{T}$, $f(z)$, $z \in \mathbb{D}$ denotes its harmonic extension at point z.

Lemma 3.1 *Let a weight w satisfies the Muckenhoupt (A_2) condition, and μ be a Carleson measure on $\mathbb{D}$. Then for any function $f \in L^2(\mathbb{T}, w)$, the following inequality holds,*

$$\iint_{\mathbb{D}} |f(z)|^2 w(z)\, d\mu(z) \le C \int_{\mathbb{T}} |f|^2 w\,,$$

where the constant C depends on Carleson norm of μ and Mackenhoupt constant of w.

Proof. By Corollary 1 it is sufficient to show that

$$\int_{\mathbb{D}} \frac{1-|s|^2}{|1-\overline{s}z|^2}[w^{-1}(z)]^2 w(z) d\mu(z) \le C \int_{\mathbb{D}} \frac{1-|s|^2}{|1-\overline{s}z|^2}[w^{-1}(z)] d\mu(z) \le C' w^{-1}(s) \tag{3.1}$$

(the first inequality trivially holds because of Muckenhoupt condition).

If we introduce a new variable $\tilde{z} = \frac{z-s}{1-\overline{s}z}$ (so $z = \frac{\tilde{z}+s}{1+\overline{s}\tilde{z}}$) and a new measure $\tilde{\mu}$

$$\tilde{\mu}(\tilde{z}(\Omega)) = \iint_{\Omega} \left| \left(\frac{z-s}{1-\overline{s}z} \right)' \right| d\mu(z) = \iint_{\mathbb{D}} \frac{1-|s|^2}{|1-\overline{s}z|^2} d\mu(z)$$

then (3.1) can be rewritten as

$$\iint_{\mathbb{D}} w^{-1} \left(\frac{\tilde{z}+s}{1+\overline{s}\tilde{z}} \right) d\tilde{\mu}(\tilde{z}) == \iint_{\mathbb{D}} \widetilde{w^{-1}}(\tilde{z}) d\tilde{\mu}(\tilde{z})$$

It is well known that if μ is Carleson, then $\tilde{\mu}$ is also Carleson, and Carleson norms of μ and $\tilde{\mu}$ are equivalent. It is also well known that a Möbius transform of a Muckenhoupt weight is again a Muckenhoupt weight with equivalent constant. So it is sufficient to show that (3.1) holds for $s = 0$.

By Cauchy and Jensen inequalities

$$\exp\{[\log w^{-1}](z)\} \le [w^{-1/2}(z)]^2 \le w^{-1}(z)\,,$$

and since the weight w^{-1} satisfies the Muckenhoupt (A_2) condition,

$$w^{-1}(z)/\exp\{[\log w^{-1}](z)\} \le C.$$

Therefore

$$w^{-1}(z) \le C[w^{-1/2}(z)]^2\,.$$

Since the measure μ is Carleson,

$$\int_{\mathbb{D}} [w^{-1}(z)] d\mu(z) \le \int_{\mathbb{D}} [w^{-1/2}(z)]^2 d\mu(z) \le \int_{\mathbb{T}} [w^{-1/2}(t)]^2 dm(t) = \int_{\mathbb{T}} [w^{-1}(t)] dm(t) = w^{-1}(0)$$

and (3.1) for $s = 0$ and hence the lemma is proved. $\square$

Theorem 3.2 *let w be a Muckenhoupt (A_2) weight, f be a function in $L^2(w)$, such that $f(0)=0$. Then there exists a constant C such that*

$$\frac{1}{C}\int_{\mathbb{T}}|f|^2 w \le \iint_{\mathbb{D}} |\nabla f(z)|^2 w(z)(1-|z|^2)\,dxdy \le C\int_{\mathbb{T}}|f|^2 w$$

Proof. Applying Green's formula and using the notation $\langle .,. \rangle$ for standard *linear* duality, $\langle a,b\rangle = \sum a_k b_k$, one can write

$$\int_{\mathbb{T}}|f|^2 w = \frac{1}{2\pi}\iint_{\mathbb{D}} \Delta\left(|f(z)|^2 w(z)\right)\log\frac{1}{|z|}\,dxdy =$$

$$= \frac{1}{2\pi}\iint_{\mathbb{D}} |\nabla f(z)|^2 w(z)\log\frac{1}{|z|}\,dxdy +$$

$$+\frac{1}{2\pi}\iint_{\mathbb{D}} f(z)\langle\nabla\overline{f}(z),\nabla w(z)\rangle\log\frac{1}{|z|}\,dxdy + \frac{1}{2\pi}\iint_{\mathbb{D}} \overline{f}(z)\langle\nabla f(z),\nabla w(z)\rangle\log\frac{1}{|z|}\,dxdy =$$

$$= \frac{1}{2\pi}\iint_{\mathbb{D}} |\nabla f(z)|^2 w(z)\log\frac{1}{|z|}\,dxdy + \frac{1}{\pi}\mathrm{Re}\left\{\iint_{\mathbb{D}} \overline{f}(z)\langle\nabla f(z),\nabla w(z)\rangle\log\frac{1}{|z|}\,dxdy\right\}$$

Since $f(0)=0$, the logarithmic singularity $\log\frac{1}{|z|}$ does not influence the integrals, so

$$\iint_{\mathbb{D}} \Delta\left(|f(z)|^2 w(z)\right)\log\frac{1}{|z|}\,dxdy \asymp \iint_{\mathbb{D}} \Delta\left(|f(z)|^2 w(z)\right)(1-|z|^2)\,dxdy$$

where $\asymp$ means equivalence in a sense of two-sided estimate.

Let us estimate

$$\left|\iint_{\mathbb{D}} \overline{f}(z)\langle\nabla f(z),\nabla w(z)\rangle\log\frac{1}{|z|}\,dxdy\right| \le C\iint_{\mathbb{D}} |f(z)|\,|\nabla f(z)|\,|\nabla w(z)|(1-|z|^2)\,dxdy \le$$

$$\le \left(\iint_{\mathbb{D}} |\nabla f(z)|^2 w(z)(1-|z|^2)\,dxdy\right)^{1/2}\left(\iint_{\mathbb{D}} |f(z)|^2 w(z)\frac{|\nabla w(z)|^2}{w(z)^2}(1-|z|^2)\,dxdy\right)^{1/2} \le$$

$$\le \left(\iint_{\mathbb{D}} |\nabla f(z)|^2 w(z)(1-|z|^2)\,dxdy\right)^{1/2}\left(\int_{\mathbb{T}}|f|^2 w\right)^{1/2}$$

(the last inequality holds by Lemma 3) If we denote

$$X = \int_{\mathbb{T}}|f|^2 w, \qquad A = \iint_{\mathbb{D}} |\nabla f(z)|^2 w(z)(1-|z|^2)\,dxdy\,,$$

then the above estimates give us $X \le C(A+(AX)^{1/2})$ and $A \le C(X+(AX)^{1/2})$. Therefore X and A are equivalent. $\square$

4. Boundedness of the Hilbert Transform.

Now we can prove that if w satisfies the Muckenhoupt (A_2) condition then the Hilbert transform (or, equivalently, the Riesz Projection P_+) bounded operator in $L^2(w)$.

Consider a weighted vector-valued L^2-space $\mathcal{H} = L^2(\mathbb{D} \to \mathbb{C}^2, w(z)(1-|z|^2)\,dxdy)$. Let $L_0^2(w) = \{f \in L^2(w) : f(0) = 0\}$. Since $|\nabla f|^2 = 2(|\partial f|^2 + |\overline{\partial} f|^2)$, Theorem 3 asserts that the mapping

$$f \mapsto \begin{pmatrix} \partial f(z) \\ \overline{\partial} f(z) \end{pmatrix}$$

gives a representation of $L_0^2(w)$ as a subspace of $\mathcal{H}$ (with equivalent norm). Here again $f(z)$, $z \in \mathbb{D}$ denotes the harmonic extension of f.

Consider an operator $\mathbb{P}_+$ on $\mathcal{H}$,

$$\mathbb{P}_+ \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

The operator $\mathbb{P}_+$ is obviously bounded, and and its restriction onto $L_0^2(w)$ is nothing but the Riesz Projection P_+. Therefore P_+ is bounded on $L_0^2(w)$. To complete the proof note that for $f \in L^2(w)$

$$P_+ f = P_+(f - f(0)) + f(0)\,.$$

The operator

$$f \mapsto f(0) \cdot 1$$

is a bounded operator in $L^2(w)$; it is an easy exercise on the Cauchy inequality that its norm is exactly $(w(0)w^{-1}(0))^{1/2}$. Since $f - f(0) \in L_0^2(w)$ the operator P_+ is bounded on $L^2(w)$. $\square$

References

[1] N. K. Nikolskii, "Treatise on the shift operator," Springer-Verlag, NY etc. 1985.