Recent advances in the similarity problem

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Based on joint works with S. Kupin and with N. Nikolski

Similarity problem.
Operator $T$ is similar to a normal if

$$T = RNR^{-1}, \quad T \text{ is normal, } R \text{ invertible}$$

Rich functional calculus: $f(T)$ is defined for any $f$ continuous on $\sigma(T)$, and

$$\|f(t)\| \leq C \sup_{z \in \sigma(T)} |f(z)|.$$ 

Necessary and sufficient condition for similarity (J. Wermer).

Linear Resolvent Growth (LRG) is the simplest necessary condition

$$\|(T - \lambda I)^{-1}\| \leq \frac{C}{\text{dist}(\lambda, \sigma(T))}.$$ 

Easy to check in applications.

LRG is not sufficient for similarity.

Example: shift $S$, $Sf = zf$ on the Hardy space $H^2$

Question: When LRG is sufficient for similarity?

Theorem (N. Nikolski, M. Benamara) Let

$\|T\| \leq 1, \text{rank}(I - T^*T) < \infty, \text{and } \sigma(T) \neq \text{clos } \mathbb{D}.$

Then LRG $\implies$ similarity to a normal operator.

One of the main ingredients (discrete spectrum) follows from a vector generalization of the Carleson Interpolation Theorem.

LRG implies that the spectral projections onto isolated points of the spectrum are uniformly bounded.

So LRG $\implies$ Uniform Minimality.

Eigenvectors—reproducing kernels
Uniform minimality $\implies$ Riesz basis.

Scalar case follows from the Carleson interpolation Theorem (Nikolskii–Pavlov, Katznelson, late 60s)
Vector case: Treil 85,

Conjecture. One can replace condition $\text{rank}(I - T^*T) < \infty$ by $I - T^*T \in \mathcal{S}_1$ ($\mathcal{S}_1$ means trace class, $\sum s_k < \infty$).

Compare with Kato–Rosemblum theorem about absolutely continuous spectrum.

The conjecture is false!

Theorem (S. Kupin, S. Treil) There exists a contraction $T$, $I - T^*T, I - TT^* \in \cap_{p>0} \mathcal{S}_p$, $\sigma(T) \neq \text{clos } \mathbb{D}$, such that $T$ satisfies LRG, but not similar to a normal operator.

Moreover, no reasonable class of operator can replace the ideal of finite rank operators.
Reasonable class: $\Phi(A) < \infty$,
$\Phi : B(H) \to \mathbb{R}_+ \cup \{\infty\}$, $\Phi(0) = 0$, and such, that

1. $\Phi$ is increasing, i.e. $\Phi(A) \leq \Phi(B)$ if $A \leq B$;
2. $\Phi(A) < \infty$ if rank $A < \infty$;
3. $\Phi$ is upper semicontinuous, i.e. if $A_n \nearrow A$ ($A_n \leq A, \|A_n - A\| \to 0$) then $\Phi(A) \leq \lim_n \Phi(A_n)$;
4. $\Phi$ is lower semicontinuous in the following weak sense: if rank $A < \infty$ and rank $A_n \leq N$, $\lim_n \|A_n\| = 0$, then $\lim_n \Phi(A \oplus A_n) = \Phi(A)$

$\Phi(A) := \Phi((A^*A)^{1/2})$ for nonselfadjoint $A$.

Examples:

1. $\Phi(A) = \|A\|_{\mathcal{S}_p} = \left(\sum s_n(A)^p\right)^{1/p}$, where $s_n(A)$ is $n$th singular value of the operator $A$.
   In this case $\Phi(A) < \infty$ means exactly $A \in \mathcal{S}_p$

2. $\Phi(A) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|A\|_{\mathcal{S}_1/n}}{1 + \|A\|_{\mathcal{S}_1/n}}$; in this case $\Phi(A) < \infty$ if and only if $A \in \bigcap_{\rho > 0} \mathcal{S}_\rho$;

3. Any weighted sum of singular numbers, for example
   $\Phi(A) = \sum_{n=1}^{\infty} 2^{2^n} s_n(A)$.

4. The function $\Phi_\psi$, $\Phi_\psi(A) := \sum_{n=0}^{\infty} \psi(s_n(A))$,
   where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing and continuous at 0 function, satisfying $\psi(0) = 0$.

Construction. Construct finite rank $A_n$ satisfying

$$\|(\lambda I - A_n)^{-1}\| \leq \frac{C}{\text{dist}(\lambda, \sigma(A_n))},$$

similar to normal

$$A_n = R_n^{-1} N_n R_n, \quad \|R_n\| \cdot \|R_n^{-1}\| \to \infty$$

Put $T_n := a_n A_n + b_n I$ and define

$$T = \bigoplus_{n=1}^{\infty} T_n$$

Picking $a_n$ small and $b_n$ close to 1, we can guarantee small defects.

Construction of $A_n$:

Take any basis $\{f_k\}_{k=1}^{\infty}$, which is not unconditional basis, and any bounded monotone sequence $a_k$.

Define $A_n$ on span$\{f_k; 1 \leq k \leq n\}$ by

$$A_n f_k = a_k f_k$$

System of exponents

$$e^{int}, \quad n = 1, 2, 3, \ldots$$

in $L^2((-\pi, \pi), |t|^\alpha dt), -1 < \alpha < 1$
Reason for the counterexample:

LRG and similarity to a normal operator are Möbius invariant, but \(\|I - T^*T\|_{\mathcal{S}_1}\) (or \(\Phi(I - T^*T)\)) is not.

Invariant trace class defects.

\[
T_\mu := (T - \mu I)(I - \mu T)^{-1}
\]

**Theorem (S. Kupin)** Let \(\|T\| \leq 1\),

\[
\sup_{\mu \in \mathbb{D}} \|I - T_\mu^*T_\mu\|_{\mathcal{S}_1} < \infty,
\]

and let \(\sigma(T) \neq \text{clos } \mathbb{D}\). Then LRG \(\implies\) similarity to normal.

**Conjecture** The above condition (1) is sharp, i.e. the condition

\[
\sup_{\mu \in \mathbb{D}} \|I - T_\mu^*T_\mu\|_{\mathcal{S}_p} < \infty, \quad p > 1
\]

does not guarantee that

LRG \(\implies\) similarity to a normal operator.

Similarity problem for non-contractions.

Let \(T := U + K\), where \(U\) is unitary,

\[
K = ba^* = (\cdot, a)b
\]

be a rank one operator.

Does LRG \(\implies\) similarity in this case?

The answer is not! (N Nikolski, S. Treil, 1999)

**Notation:** Vector \(b\) is treated as an operator,

\[
b : \mathbb{C} \to H, \quad \alpha \mapsto \alpha b,
\]

\(b^*\) is its adjoint,

\[
b^* : H \to \mathbb{C}, \quad b^*x := (x, b).
\]

**Resolvent:**

\[
R_T^\lambda := (\lambda I - T)^{-1}
\]
Resolvent of rank one perturbation

\[ \lambda I - T = \lambda I - U - K = (\lambda I - U)(I - (\lambda I - U)^{-1}K) = (\lambda I - U)(I - R^U_K) \]

Lemma  Let \( K = ba^* = (\cdot, a)b \). Then

\[ (I - K)^{-1} = I + \frac{1}{d}K, \]

where \( d = \det(I - K) = 1 - (b, a) \)

Inverting rank one perturbation \( (I - R^U_K) \) of \( I \) we get

\[ R^T = (\lambda I - T)^{-1} = R^U_K + \frac{1}{d(\lambda)}R^U_K R^U_K, \]

where \( d(\lambda) = \det(I - b\lambda a^*) = 1 - (b\lambda, a) \)

\[ b\lambda := R^U_K b := (\lambda I - U)^{-1}b \]

So

\[ d(\lambda) = 1 - \int_T \frac{\overline{a}b}{\lambda - z} d\mu(z). \]

When \( T \) satisfies LRG

\[ R^T = (\lambda I - T)^{-1} = R^U_K + \frac{1}{d(\lambda)}R^U_K R^U_K. \]

\( R^U_K \) satisfies LRG, study

\[ R^U_K R^U_K = b\lambda a^*_\lambda, \quad b\lambda := R^U_K b, \quad a\lambda = (R^U_K)^*a \]

In spectral representation

\[ b\lambda(z) = b(z)/(\lambda - z), \quad a\lambda(z) = a(z)/(\lambda - z), \]

Since \( \|R^U_K R^U_K\| = \|b\lambda\| \cdot \|a\lambda\| \) and

\[ \|a\lambda\|^2 = \int_T \frac{|a(z)|^2}{|z - \lambda|^2} dm(z) = \int_T \frac{|a(z)|^2}{|1 - \overline{\lambda}z|^2} dm(z) \]

\[ \|b\lambda\|^2 = \int_T \frac{|b(z)|^2}{|z - \lambda|^2} dm(z) = \int_T \frac{|b(z)|^2}{|1 - \overline{\lambda}z|^2} dm(z) \]

(Poisson extensions of \( |a|^2 \) and \( |b|^2 \) up to the multiplication by \( 1 - |\lambda|^2 \)).

Let us consider the simplest case:

\[ U \text{ be multiplication by } z \text{ in } L^2(\mathbb{T}). \]

Assume that \( ab \equiv 0 \) on \( \mathbb{T} \), so \( (b\lambda, a) = 0 \forall \lambda \).

Then \( d(\lambda) \equiv 1 \).

Since \( \sigma(T) = \sigma(U) = \mathbb{T} \),

\[ \text{dist}(\lambda, \sigma(T)) \asymp 1 - |\lambda|^2. \]

LRG is equivalent to

\[ \mathcal{P}_\lambda(|a|^2) \cdot \mathcal{P}_\lambda(|b|^2) \leq C, \quad \lambda \in \mathbb{D} \]

(Sarason’s condition).
$T$ is similar to a unitary operator iff
\[ (R^T f, g) = \int_{\mathbb{T}} \frac{d\mu(z)}{\lambda - z} = -[C\mu](\lambda) \]
where $\mu = \mu_{f,g}$ is a complex measure, 
$\var\mu_{f,g} \leq C \cdot \|f\| \cdot \|g\|$.

For a measure $\mu$,
\[
[C\mu](\lambda) = \begin{cases} 
\sum_{n \geq 0} \hat{\mu}(n+1)\lambda^n, & |\lambda| < 1 \\
-\sum_{n < 0} \hat{\mu}(n+1)\lambda^n, & |\lambda| > 1
\end{cases}
\]
Therefore for $|\lambda| < 1$ we have
\[
[C\mu]^H(\lambda) := [C\mu](\lambda) - [C\mu](1/\lambda) = P_{\lambda}(\pi d\mu(z))
\]
So, the harmonization $[C\mu]^H$ of $C\mu$ is the Poisson extension of a (complex) measure of bounded variation.

But Sarason’s condition
\[
P_{\lambda}(|a|^2) \cdot P_{\lambda}(|b|^2) \leq C, \quad \lambda \in \mathbb{D},
\]
doesn’t imply that $M_a P_+ M_b$ is bounded (F. Nazarov)

Using Nazarov’s counterexample one can construct $a$, $b$ satisfying Sarason’s condition, $ab \equiv 0$, and $f$, $g \in L^2$ such that
\[
\int_{\mathbb{T}} | \Phi^H rz | dm(z) \to \infty \quad \text{as } r \to 1, r < 1,
\]
so $\Phi^H$ is not a harmonic extension of a complex measure of finite variation.

To check similarity for $T = U + K$ enough to check the harmonization of $\Phi$,
\[
\Phi(\lambda) = (R^{\lambda} f, g) = \int_{\mathbb{T}} \frac{\tilde{a}f dm(z)}{\lambda - z} \cdot \int_{\mathbb{T}} \frac{\tilde{b}g dm(z)}{\lambda - z} = \{C\tilde{a}f\}(\lambda) \cdot \{C\tilde{b}g\}(\lambda).
\]

Therefore
\[
\Phi^H = (P_+ \tilde{a}f) \cdot (P_+ \tilde{b}g) - (P_- \tilde{a}f) \cdot (P_- \tilde{b}g)
\]
The boundary values of $\Phi^H$ are in $L^1(\mathbb{T}) \forall f, g$ (with uniform estimates of the norm) if and only if the operator $M_a P_+ M_b$ is bounded in $L^2$.

\[
(P_+ af) \cdot (P_+ bg) - (P_- af) \cdot (P_- bg) = (P_+ af) \cdot (P_- bg) + (P_+ af) \cdot (P_+ bg) - (P_- af) \cdot (P_+ bg) - (P_- af) \cdot (P_- bg) \]

Using Nazarov’s counterexample get: an ark $\mathcal{I}$, smooth $a$, $b$ supported by $\mathcal{I}$, smooth $f$, $g$.
\[
\|f\|_2 = \|g\|_2 = 1 \text{ such that} \int_{\mathcal{I}} |bg \cdot P_+(af)| \text{ is big}
\]
Replacing $g$ by $z^N g$ make $P_-(af) \cdot P_-(bg)$ small and
\[
\int_{\mathcal{I}} | P_+(bg) \cdot P_+(af) | \text{ big}
\]
\[
\int_{\mathcal{I}} \cdots \text{ is also big.}
\]
$\eta$ — indicator of a half-circle
Replacing
\[
a \mapsto a_n := \eta(z^n) \cdot a \\
b \mapsto b_n := (1 - \eta(z^n)) \cdot b
\]
make $ab = 0$.
\[
a_n \to a/2, \quad b_n \to b/2 \quad \text{weakly in } L^2
\]
so $\int_{\mathcal{I}} \cdots \text{ is still big.}$
Problems:

1. Sufficient conditions.
   Study the bilinear operator
   \[ \{f, g\} \mapsto (P_+ af) \cdot (P_+ bg) - (P_- af) \cdot (P_- bg) \]
   Can we get anything beyond two-weight estimates for \( P_+ \)?

2. General case, \( ab \neq 0 \):
   \[ \{f, g\} \mapsto \frac{(P_+ af) \cdot (P_+ bg)}{d_+} - \frac{(P_- af) \cdot (P_- bg)}{d_-} \]

3. What to do with eigenvalues outside \( \mathbb{T} \)? Does LRG imply that the system of eigenvectors is a Riesz basis?

4. Is the condition
   \[ \sup_{z \in \mathbb{D}} \| I - T_z^* T_z \|_{S_1} < \infty \]
   sharp?