Control theory methods in harmonic analysis and vice versa

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Plan.

• Introduction: how did I got myself into control

• What is this talk is not about:
  - Not about $H^\infty$ optimization
  - Not about frequency domain methods
  - Not about any well known connection between Harmonic Analysis and Control Theory

• Part one: Bellman function in Harmonic Analysis
  - Bellman equation in stochastic control
  - Application to Harmonic Analysis
  - Interpretations
  - Conclusions

• Part two: Application of singular integrals in Control. The problem of upper bound for the structured singular value $\mu$.
  - Statement of the problem
  - Solution via infinite-dimensional case
  - How the singular integrals appear
  - Conclusion, open problems

Bellman equation for stochastic optimal control

$$x^t = x + \int_0^t \sigma(\alpha^s, x^s) \, dw^s + \int_0^t b(\alpha^s, x^s) \, ds$$

$t$ is the time, $w^t$ is a $d_1$-dimensional Wiener process (white noise), $\sigma = \sigma(\alpha, y)$ is a $d \times d_1$ matrix, and $b$ is a $d$-dimensional vector. The process $\alpha^t$ is a control that we have to chose. We denote by $A \subset \mathbb{R}^{d_2}$ the set of admissible controls.

Want to maximize

$$v^\alpha(x) = \mathbb{E} \int_0^\infty f^{\alpha^t}(x^t) \, dt + \lim_{t \to \infty} \mathbb{E}(F(x^t))$$

The Bellman function for stochastic control: $\boxed{v(x) = \sup_\alpha v^\alpha(x)}$.

The Bellman equation for the stochastic optimal control is based on two things:

1. Bellman’s principle,
2. Ito’s formula.
Bellman’s principle: \[ v(x) = \sup_{\alpha} \mathbb{E} \left[ \int_0^t f^{\alpha^*}(x_s) \, ds + v(x^t) \right] \] for each \( t \geq 0 \).

Ito’s formula:

First term in the Taylor expansion of \( v(x^{s+\Delta s}) - v(x^s) \):

\[
\mathbb{E} \left[ \sum_{k=1}^d \frac{\partial v}{\partial x_k}(x^s) \sigma_{k,j}(\alpha, x^s) \Delta w^s_j + \sum_{k=1}^d \frac{\partial v}{\partial x_k}(x^s) b_k(\alpha, x^s) \Delta s \right].
\]

Since \( \mathbb{E} \Delta w^s_k \Delta w^s_m = \delta_{k,m} \Delta s \), we need the second term

\[
\frac{1}{2} \sum_{i,j} \frac{\partial^2 v}{\partial x_i \partial x_j} \left( \sum_k \sigma^{jk} \Delta w^s_k + b_j(\alpha, x^s) \Delta s \right) \cdot \left( \sum_k \sigma^{ik} \Delta w^s_k + b_i(\alpha, x^s) \Delta s \right)
\]

Bellman equation:

\[
\sup_{\alpha \in A} [\mathcal{L}^{\alpha}(x)v(x) + f^{\alpha}(x)] = 0
\]

where

\[
\mathcal{L}^{\alpha}(x) := \sum_{i,j=1}^d a^{ij}(\alpha, x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{k=1}^d b_k(\alpha, x) \frac{\partial}{\partial x_k},
\]

\[
a^{ij}(\alpha, x) := \frac{1}{2} \sum_{k=1}^d \sigma^{ik}(\alpha, x) \sigma^{jk}(\alpha, x).
\]

Recall

\[
x^t = x + \int_0^t \sigma(\alpha, x^s) \, dw^s + \int_0^t b(\alpha, x^s) \, ds
\]

- Existence??
- Uniqueness??
- Necessary??
- Sufficient??

It is better to consider supersolutions

\[
\sup_{\alpha \in A} [\mathcal{L}^{\alpha}(x)v(x) + f^{\alpha}(x)] \leq 0
\]

In problems we consider often \( \sigma = (\alpha_1, \alpha_1, \ldots, \alpha_d)^T \).

The second order part of the equation will be

\[
\frac{1}{2} \sum_{i,j} \alpha_i \alpha_j \frac{\partial^2 v}{\partial x_i \partial x_j} \quad \text{(concavity)}
\]

A problem in Harmonic Analysis: Carleson Embedding Theorem.

\( \mathbb{R}_+^2 = \{(x, y) : x \in \mathbb{R}, y > 0\} \) upper half plane; \( \mu \) measure in \( \mathbb{R}_+^2 \);

\( \mathcal{H}f \) denotes harmonic extension to \( \mathbb{R}_+^2 \) of a function \( f \in L^2(\mathbb{R}) \)

\( Q_I \) — a square with the base \( I \), \( Q_I = \{(x, y) \in \mathbb{R}_+^2 : x \in I, 0 < y < |I|\} \) (Carleson square);

Theorem. The embedding

\[
\int_{\mathbb{R}_+^2} |\mathcal{H}f|^2 d\mu \leq C \int_{\mathbb{R}} |f(x)|^2 dx
\]

holds if and only if \( \mu(Q_I) \leq C|I| \) for all intervals \( I \).

Not hard, but very important theorem.

Let us treat its dyadic analogue as a control problem

Dyadic—replace Poisson averages by averages over dyadic intervals.
Bellman Function. Define averages:

\[ F_I := \frac{1}{|I|} \int_I f^2 = F, \quad f_I := \frac{1}{|I|} \int_I f \quad M_I = \frac{1}{|I|} \sum_{J \subseteq I} \mu_J, \]

Fix an interval \( I \in \mathcal{D} \), and numbers \( f, F, M \geq 0 \).

Consider all \( f \in L^2(\mathbb{R}) \) and sequences \( \{\mu_J\}_{J \in \mathcal{D}} \) such that

\[ M_J \leq 1 \quad \forall J \in \mathcal{D}, \quad M_I = M \]

\[ f_I = f, \quad F_I = F. \]

Define

\[ B(F, f, M) = B_I(F, f, M) = \frac{1}{|I|} \sup \left\{ \sum_{J \subset I} f_J^2 \mu_J : \mu_J, f \text{ satisfy above constrains} \right\} \]

Does not depend on \( I \) because of scaling.

The answer is given by the famous Carleson embedding theorem:

**Theorem**

\[ \sum_{J \in \mathcal{D}} \mu_J f_J^2 \leq C \|f\|_2^2 \quad \forall f \in L^2(\mathbb{R}) : \]

\[ \Leftrightarrow \sum_{J \text{ is dyadic}} \mu_J \leq C'|I| \quad \text{for any dyadic interval } I, \]

Carleson measure condition.

The known proofs are not to hard, but the theorem is very important in harmonic analysis.

A naïve approach — let us pretend that we do not know any standard technique.

**Properties of \( B \):**

1. **Domain:** \( f^2 \leq F, \ 0 \leq M \leq 1 \) (Cauchy inequality and the Carleson condition);
2. **Range:** \( 0 \leq B(F, f, M) \leq CF \) (our belief that the theorem is true);
3. **Key property:** for all \( 0 \leq m \leq M \)

\[ B(F, f, M) \geq mf^2 + \frac{1}{2} \left\{ B(F^+, f^+, M^+) + B(F^-, f^-, M^-) \right\}, \]

whenever

\[ F = \frac{1}{2}(F^+ + F^-), \quad f = \frac{1}{2}(f^+ + f^-), \quad \text{and } M = \frac{1}{2}(M^+ + M^-) + m. \]

**Explanation of 3:** \( m = \mu_I / |I| \), gain \( B(F^+, f^+, M^+) + B(F^-, f^-, M^-) \) summing over \( I^+ \) and \( I^- \).
If we find any $B$ satisfying 1–3, we are done!

$$\int_\Omega |I| \cdot B \leq |I| \cdot B_\mu - |I_+| \cdot B_+ \mu_I \cdot (f_I)^2 - |I_-| \cdot B_+ \mu_I \cdot (f_I)^2$$

The key property can be rewritten as

$$\mu_I f_I^2 \leq |I| \cdot B(F_I, f_I, M_I)$$

Summing over all subintervals of $I$ we get

$$\sum_{J \subset I} f_J^2 \mu_J \leq |I| \cdot B(F_I, f_I, M_I) - (\geq 0) \leq C F_I = C \int_I f^2 dx.$$

Differential form of the key property 3.

3. $B(F, f, M) \geq m f^2 + \frac{1}{2} \left\{ B(F_+, f_+, M_+) + B(F_-, f_-, M_-) \right\}$ $\iff$

3'. $d^2 B \leq 0$;

3''. $\frac{\partial B}{\partial M} \geq f^2$.

**Alternative explanation:** The vector $\alpha$ of controls is determined by

$$\alpha_1 = (F_I - F_I), \quad \alpha_2 = (f_I - f_I), \quad \alpha_3 = (M_I - M_I)/2, \quad \alpha_4 = -|I|^{-1} \mu_I$$

Bellman equation for the continuous analogue of the process:

$$\sup_\alpha \left\{ \frac{1}{2} \sum_{j,k=1}^3 \frac{\partial^2 B}{\partial x_j \partial x_k} \alpha_j \alpha_k + \frac{\partial B}{\partial M} \alpha_4 - \alpha_4 f^2 \right\} = 0.$$

Any solution of the inequality solves the equation ($\alpha_k = 0$)!

So, to prove the Carleson embedding theorem we need to find a function $B(F, f, M)$ on $f^2 \leq F$, $0 \leq M \leq 1$, satisfying

2. $0 \leq B(F, f, M) \leq C F$

3'. $d^2 B \leq 0$;

3''. $\frac{\partial B}{\partial M} \geq f^2$.

**An answer:**

$$B(F, f, M) = 4 \left( F - \frac{f^2}{1 + M} \right)$$

(the constant 4 is sharp).

**Remarks**

- Bellman function method work on any martingale;
- It changes proof and counterexample;
Another example:

\[ T \text{ is Hilbert Transform, } \quad Tf(s) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{s - t} dt. \]

When \( T \) is bounded in \( L^2(w) \)?

\[ \|f\|_{L^2(w)}^2 = \int_{\mathbb{R}} |f(x)|^2 w(x) dx. \]

Hunt–Muckenhoupt–Wheeden Theorem. \( P_+ \) is bounded in \( L^2(w) \) iff

\[ \sup_I \left( \frac{1}{|I|} \int_I w \right) \left( \frac{1}{|I|} \int_I w^{-1} \right) < \infty \quad (A_2) \]

**The Bellman function:**

\[ B(f, g, w, u) = C \cdot (F + G) - \left( \frac{f^2}{u} + \frac{g^2}{w} \right) \left( 1 + \varepsilon \cdot (wu)^{-\alpha} \right), \]

where \( \alpha > 1 \) and \( \varepsilon > 0 \) is sufficiently small. **The main inequality**

\[ -d^2 B \geq \delta \left( \frac{(df)^2}{u} + \frac{(dg)^2}{w} \right) \geq \delta |df| \cdot |dg| \]

(inequality for quadratic forms of \( df, dg, du, dw \)).

**Part 2: Singular integrals in robust control.**

- Changes proof and counterexample: a single function gives a proof, but if we prove that such function does not exist, we get a counterexample (implicitly).
- Bellman function need not to be smooth but it is not a problem: take convolution with appropriate smooth mollifier. Need to show that such operation does not change finite difference inequalities, but that is usually not a problem.

**Some conclusions**

- Bellman equation is in fact inequality
- Usually some kind of concavity condition
- Often we only need to find a suboptimal solution
- Possible to understand what the equation for the optimal solution should be, and sometimes it is possible to solve it. Extra work is necessary for justification
- Reducing number of variables using homogeneity and eliminating “non-essential” variables
- Domain is very essential!
- To get from a dyadic analogue to the “real” problem, one usually plugs Bellman function in some Green’s formula. Now it is more art than science.
- Construct Bellman function from known “building blocks”
Statement of the problem: Let $A$ be $n \times n$ matrix.

The structured singular value $\mu = \mu(A)$:

$$
\mu(A) : = \inf \{ \|\Delta\| : \Delta \text{ diagonal, s.t. } (I - A\Delta) \text{ not invertible} \}
$$

$$
= \inf \{ \|\Delta\| : \Delta \text{ diagonal, s.t. } (I - A\Delta) \text{ not invertible} \}
$$

Here and below, $\|A\|$ always denotes the induced (by the Euclidean norm in $\mathbb{C}^n$) operator norm of $A$, i.e. its maximal singular value.

Entries of the matrix $\Delta$ are complex, so we deal with complex structured singular value.

$A\Delta$ can be replaced by $\Delta A$ — a simple exercise in linear algebra.

$\mu$ appears in connection with robust control.

Trivial estimates: $\rho(A) \leq \mu(A) \leq \|A\|$.

Crash course in robust stability:

Figure 1: Uncertainty $\Delta$ in the feedback loop. Assume that $\|\Delta\| < \rho$.

Closed loop system

$$
\begin{cases}
  y = Gu \\
  u = \Delta y
\end{cases}
$$

Here $G$, $\Delta$ are bounded linear, causal, time invariant operators on $\ell^2(\mathbb{R}^n)$, or, after Fourier transform multiplications by $n \times n$ matrix functions $G \in H^\infty$ on $H^2(\mathbb{C}^n)$ ($H^2$ with values in $\mathbb{C}^n$).

The system is stable

$$
\begin{cases}
  y = Gu + v_1 \\
  u = \Delta y + v_2
\end{cases}
$$

if the mapping $(v_1, v_2) \mapsto (y, u)$ is bounded

$\iff$

$$
(I_{n \times n} - G\Delta)^{-1} \in H^\infty
$$

$\iff$

$$
(I - G\Delta)^{-1} \in H^\infty.
$$

Explanation:

$$
\begin{pmatrix}
  I & -G \\
  -\Delta & I
\end{pmatrix}
\begin{pmatrix}
  y \\
  u
\end{pmatrix}
= \begin{pmatrix}
  v_1 \\
  v_2
\end{pmatrix}
$$

Let us recall that for the multiplication operator $M_G$, $\|M_G\| = \|G\|_\infty$. Here for a matrix function $G \in H^\infty$,

$$
\|G\|_\infty = \text{esssup}_{\theta} \sigma_{\text{max}}(G(e^{i\theta}))
$$

(For a matrix $A$, $\sigma_{\text{max}}(A)^2 = \text{maximal eigenvalue of } A^*A$)

Stability margin $\rho$:

$$
\max \; r : \text{the system is stable for any } \Delta, \quad ||\Delta||_\infty < r
$$

$$
\inf \; R : \text{the system is unstable for some } \Delta, \quad ||\Delta|| = R
$$

Small Gain Theorem. $\rho = \|M_G\|^{-1} = \|G\|_\infty^{-1}$.

If $\|\Delta\|_\infty < \|G\|_\infty^{-1}$ then $I - G\Delta$ is clearly invertible.
Usually disturbance $\Delta$ has some structure, e.g. $\Delta$ is diagonal.

\[ \begin{array}{c}
\Delta_1 & W_1 \\
\hline
G & \Delta_2 & W_2 \\
\hline
K & \Delta
\end{array} \]

Figure 2: Feedback stabilization with uncertainties in observation and control channels.

This configuration can be reduced to the standard form with $\Delta = \text{diag}(\Delta_1, \Delta_2)$.

Structured Small Gain Theorem. For structured (diagonal) LTI perturbations $\Delta$

\[
\text{stability margin} = \frac{1}{\text{esssup}_{\theta \in (-\pi, \pi)} \mu(G(e^{i\theta}))}
\]

General structured norms. $\mathbb{U}$ — class of uncertainties $\Delta$.

\[
\text{SN}_U(G) := \left( \inf_{\Delta \in \mathbb{U}} \{ \| \Delta \| : 0 \in \sigma(I - G\Delta) \} \right)^{-1}
\]

In our case $\mathbb{U}$ — diagonal, LTI.

In this case $\text{SN}_U(G) = \text{esssup}_{\theta \in (-\pi, \pi)} \mu(G(e^{i\theta}))$

Other choices: diagonal LTV, or NLTI, or NLTV.

Clearly, stability margin $\rho := 1/\text{SN}_U(G)$.

$\mu$ is hard to compute.

Need a “good” upper bound.

\[ \mu(A) \leq \|A\| \]

but $\|A\|$ is not a good bound:

For $A = \begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix}$, $R$ — big

\[ \|A\| \sim R \quad \text{but} \quad \mu(A) = 1. \]

Upper bound $\overline{\rho}$: Know $\rho(A) \leq \mu(A) \leq \|A\|$.

If $D$ is diagonal, invertible, then $\mu(DAD^{-1}) = \mu(A)$:

\[ I - DAD^{-1}\Delta = D(I - A\Delta)D^{-1} \]

Therefore

\[ \mu(A) \leq \inf \{ \|DAD^{-1}\| : D \text{ diagonal, invertible} \} \]

— The infimum is the upper bound $\overline{\rho}$.

The upper bound $\overline{\rho} = \inf \{ \|DAD^{-1}\| \}$ is easier to compute: the computation reduces to a convex optimization problem (solving Linear Matrix Inequalities (LMI)).

\[ \overline{\rho}(A) \leq a \iff \exists \exists D = D^*, \text{diagonal, } \|DAD^{-1}\| \leq a + \varepsilon \]

\[ \iff D^{-1}A^*D^2AD^{-1} \leq (a + \varepsilon)^2I \iff A^*D^2A \leq (a + \varepsilon)^2D^2 \]

Linear in $D^2$. 

Back to the example:

\[
A = \begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix}, \quad R \text{ big}
\]

For \( D = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \), \( \varepsilon \to 0 \)

\[
DAD^{-1} = \begin{pmatrix} 1 & \varepsilon R \\ 0 & 1 \end{pmatrix}
\]

so

\[ \bar{\mu}(A) \leq 1. \]

But

\[ 1 = \rho(A) \leq \mu(A) \leq \bar{\mu}(A) \leq 1, \quad \text{so} \quad \mu(A) = 1. \]

Doyle, 1982: \( \bar{\mu} = \mu \) for \( n \leq 3 \).

Also known: For \( n \geq 4 \), \( \exists A \) such that \( \mu(A) < \bar{\mu}(A) \)

**S-procedure:** let \( \sigma_k \) be quadratic forms, such that

\[
\sigma_k[x] \geq 0, \quad \forall k = 1, \ldots, n \implies \sigma_0[x] \geq 0,
\]

and such that

\[
\exists x_0, \quad \sigma_k[x_0] > 0, \quad k = 1, \ldots, n.
\]

If there exist \( \tau_k \geq 0 \) such that

\[
\sigma_0[x] - \sum_{k=1}^{n} \tau_k \sigma_k[x] \geq 0 \quad \forall x?
\]

Yes, if \( n = 1 \) for forms on real space, and if \( n = 2 \) for forms on complex space.

**Conjecture (Doyle):** \( \exists C (= 2?) \) such that

\[ \bar{\mu}(A) \leq C\mu(A) \quad \forall A. \]

In numerical experiments \( \frac{\bar{\mu}(A)}{\mu(A)} \leq 1.3 \)

Upper bounds for other structured norms.

Consider class \( \mathcal{D} \) of *scaling operators* \( D \).

Need \( \forall D \in \mathcal{D} \) and \( \forall \Delta \in \mathcal{U} \)

\[
D\Delta D^{-1} \in \mathcal{U}, \quad \|D\Delta D^{-1}\| = \|D\|
\]

For diagonal, LTI uncertainties \( \mathcal{D} \) consists of (multiplications by) diagonal matrix functions (in frequency domain).

For diagonal LTV (or NLTI, or NLTV), \( \mathcal{D} \) consists of (multiplications by) constant diagonal matrices.
Clearly
\[ \mathcal{S} U(G) \leq \mathcal{S} N_G(G) := \inf_{D \in \mathcal{D}} \| DGD^{-1} \|. \]

Consider the singular integral operator (Cauchy integral) \( S = S_T \) defined by
\[
(Sf)(t) = \frac{1}{\pi i} \text{p.v.} \int_{\Gamma} \frac{f(s)}{s-t} ds
\]

The structure of \( S \) is well known. Let
\[
P = \frac{1}{2}(I + S), \quad Q = \frac{1}{2}(I - S).
\]

Then \( S = P - Q \) and \( P, Q \) are complimentary projections (\( P + Q = I, P^2 = P, Q^2 = Q \)), not necessarily orthogonal.

The projections \( P \) and \( Q \) are projections onto subspaces \( L^2_+ = L^2(\Gamma)_+ \), \( L^2_- := \text{clos}_{L^2} L^2 \text{Rat} \pm \);

here \( \text{Rat}_+ \) denotes all rational functions with poles outside of \( \Gamma \), and \( \text{Rat}_- \) consists of all rational functions \( f, f(\infty) = 0 \) with poles inside \( \Gamma \).

If \( \Gamma \) is the unit circle, the spaces \( L^2_\pm \) coincide with classical Hardy spaces \( H^2_\pm \), and projections \( P \) and \( Q \) are orthogonal.

\[
\Gamma = \mathbb{R}: \text{the projections } P \text{ and } Q \text{ are orthogonal projections onto the Hardy spaces in the upper and lower half planes respectively.}
\]

Since the projections \( P \) and \( Q \) are orthogonal, \( \| S_{\mathbb{R}} \| = 1 \).

\[ \mu(S_T) \leq 1. \]

**Main result:**

**Theorem.** There is a sequence of square matrices \( A_n \ (N_n \times N_n, N_n \to \infty) \)
such that
\[
\lim_{n \to \infty} \frac{\overline{\mu}(A_n)}{\mu(A_n)} = \infty.
\]

Infinitesimal analog of the main result.

Let \( L^2 = L^2(\nu) \). For an operator \( A \) on \( L^2(\nu) \) define it structured singular value \( \mu(A) \) as
\[
\mu(A) := \inf_{\psi} \{ \| M_{\psi}A^{-1} \psi \| : \psi \in L^\infty(\mu) \};
\]

here \( M_{\psi}f = \psi f \).

Define the upper bound \( \overline{\mu}(A) \) by
\[
\overline{\mu}(A) := \inf_{\psi} \{ \| M_{\psi}AM_{\psi^{-1}} \| : \psi, \psi^{-1} \in L^\infty(\mu) \};
\]

Construct \( \nu \) and \( A \) on \( L^2(\nu) \) such that \( \overline{\mu}(A) / \mu(A) \) is large.

\[
(\mathcal{S}f)(t) = \frac{1}{\pi i} \text{p.v.} \int_{\Gamma} \frac{f(s)}{s-t} ds
\]

Theorem

Let \( \Gamma \) be a simple \( C^2 \)-smooth closed curve in \( \mathbb{C} \). Let \( \varphi \in L^\infty(\Gamma), \| \varphi \|_\infty < 1 \). Then the operator \( I - \varphi S_T \) is invertible. (And therefore \( \mu(S_T) \leq 1 \)).

- Well known fact.
- Trivial for \( \Gamma = \mathbb{R} \) or \( \Gamma = \mathbb{T} \) (because \( \| S_{\mathbb{R}} \| = \| S_T \| = 1 \)).
- Invertibility (modulo compact operators) is a local property for singular integral operators, so it is Fredholm.
- No index (winding number)
Proof of the theorem about invertibility.

\[ I - \varphi S = P + Q - \varphi (P - Q) = (1 - \varphi)P + (1 + \varphi)Q \]

\[ = (1 + \varphi) \left( \frac{1 - \varphi}{1 + \varphi} P + Q \right). \]

\( I - \varphi S \) invertible iff \( aP + Q \) is invertible, where \( a = \frac{1 - \varphi}{1 + \varphi}. \)

\( \|\varphi\|_\infty < 1 \implies a, a^{-1} \in L^\infty, \) range \( a \) is in a sector with vertex at the origin and the opening angle strictly less than \( \pi \) (\( a \) is sectorial).

\[ aP + Q \] is invertible iff \( a \) is invertible, where \( a = \frac{1 - \varphi}{1 + \varphi}. \)

Standard fact: for such \( a \) operator \( aP + Q \) is invertible.

Explanation: Need to solve: \( aPf + Qf = g. \)

Denoting \( f_+ = Pf, f_- := Qf \) we get \( af_+ + f_- = g_+ + g_- \)

Applying \( P \) and \( Q \) to the equation we get \( P(af_+) = Pg, \) and \( f_- := Qg - Q(af_+). \)

Solve the first equation for \( f_+, \) then find \( f_- \), and put \( f = f_+ + f_- \).

So, \( aP + Q \) is invertible iff Toeplitz operator \( T_a \) (on Range \( P \))

\[ T_a f = P(af), \quad f \in \text{Range } P \]

is invertible

- Invertible for \( \mathbb{T} \) or \( \mathbb{R}; \)
- General \( C^2 \)-curve behave locally like \( \mathbb{R} \) (or \( \mathbb{T} \)).
- Localization theory for SIO: locally invertible operator is a Fredholm operator

Why \( \overline{\mu} \) is big:

Want to show \( \inf \{ \|\psi S \psi^{-1}\| : \psi, \psi^{-1} \in L^\infty \} \) is big.

For \( \Gamma \) let us define its Ahlfors constant \( A(\Gamma) \) by

\[ A(\Gamma) = \sup \frac{|B_r(x) \cap \Gamma|}{r}; \]

supremum is taken over all discs \( B_r(x) = \{ z \in \mathbb{C} : |z - x| < r \}, \) and \( |...| \) denotes the length of the set ....

Lemma. For any \( \psi \in L^\infty(\Gamma) \) with \( \psi^{-1} \in L^\infty(\Gamma) \) we have

\[ \|M_\psi S_T M_\psi^{-1}\| \geq c \cdot A(\Gamma), \]

where \( c > 0 \) is an absolute constant.

This Lemma immediately implies that one can construct a \( C^2 \)-smooth contour \( \Gamma \) such that \( \frac{\overline{\mu}(S_T)}{\mu(S_T)} \) is as large as we want.
Indeed, we got that $\mu(S_G) \leq 1$.

On the other hand, the above lemma implies that $\pi(\Gamma) \geq cA(\Gamma)$, where $c$ is the absolute constant from the lemma.

And it is very easy to construct a $C^2$-smooth contour $\Gamma$ with arbitrary large Ahlfors constant $A(\Gamma)$.

To estimate $S = \tilde{S}$

$$(\tilde{S}f)(t) = \frac{1}{\pi} \text{ p.v.} \int_{\Gamma} \frac{f(s)}{s-t} |ds|$$

$$\|\tilde{S}\| \geq cA(\Gamma)$$

$$\|M_{\psi}S_{\psi}^{-1}\| = \text{norm of } \tilde{S} \text{ in } L^2(w), \ w = |\psi|^2.$$  

\[ \text{norm of } \tilde{S} \text{ in } L^2(w) = \text{norm of } \tilde{S} \text{ in } L^2(w^{-1}) \quad \text{(duality)} \]

Interpolating between $L^2(w)$ and $L^2(w^{-1})$:

$$cA(\Gamma) \leq \|\tilde{S}\| \leq \|M_{\psi}S_{\psi}^{-1}\|.$$

**Discretization.** Partitions $P_n$.

Let $P_n$ be the averaging operators,

$$P_n f = \sum_{k=1}^{N(n)} \left( \frac{1}{|\Delta^n_k|} \int_{\Delta^n_k} f(z) \, |dz| \right) \chi_{\Delta^n_k},$$

where $\Delta^n_k, k = 1, 2, ..., N(n)$ are the arcs of the partition $P_n$.

Clearly, $P_n$ are orthogonal projections on $L^2(\Gamma)$. Let $X_n := \text{Range } P_n = P_nL^2.$

Define operators $S_n$ by

$$S_n = P_nS \mid X_n.$$  

Easy to show that

$$\lim_{n} \left( \inf \{ \|DS_nD^{-1}\| : D \text{ is diagonal} \} \right) \geq cA(\Gamma).$$

Show that there exists $N > 0$ such that for all $n \geq N$ and all $\varphi_n \in X_n \cap L^\infty$, $\|\varphi_n\|_{\infty} \leq 0.1$ the operators $I_n - \varphi_nS_n$ are invertible (i.e. $\mu < 10$).
Suppose, that is not true, i.e. suppose that \( \mu \geq 10 \). Suppose there exist \( \varphi_n \in X_n \cap L^\infty \), \( \| \varphi_n \|_\infty \leq 0.1 \) such that \( 0 \in \sigma(I_n - \varphi_n S_n) \).

Then there exist vectors \( f_n \in X_n, \| f_n \| = 1 \) such that \( f_n = \varphi_n S_n f_n \), so

\[
|f_n(t)| \leq 0.1 \cdot |(S_n f_n)(t)| \quad \forall t \in \Gamma.
\]

Show that this is impossible.

**Localization technique** for finite section methods for Singular Integral Equations.

One could apply standard results, if \( \varphi_n \to \varphi \) strongly.

**Main difficulty:** If \( \varphi_n \to \varphi \), \( S f_n \to S f \), \( \varphi_n S f_n \to \varphi S f \), even if we have convergence of \( \varphi_n S f_n \). Example: \( S f_n = z^n \), \( \varphi_n = z^{-n} \) in \( L^2(\mathbb{T}) \).

Have to push everything to the limit.

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**Technical Lemmas:**

For a set \( \Delta \) let \( E(\Delta) \) denote the operator of multiplication by \( \chi_\Delta \):

\[ E(\Delta) f := \chi_\Delta f. \]

**Localization Principle.** Suppose that \( \Delta_1, \Delta_2 \subset \Gamma, \text{dist}(\Delta_1, \Delta_2) > 0 \). Then the operator \( E(\Delta_1) S E(\Delta_2) \) is compact.

**Local Norm Lemma.** For any point \( \tau \in \Gamma \) and any sequence of arcs \( \Delta_n \ni \tau \) such that \( |\Delta_n| \to 0 \) we have

\[
\limsup_{n \to \infty} \| E(\Delta_n) S E(\Delta_n) \| \leq 1
\]

The proof is based on the fact that the operator behave \( S \) is locally almost as the operator \( S_{\mathbb{R}} \), which has norm 1.

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The essential norm \( \| A \|_{\text{ess}} \) of an operator \( A \) is the distance to the compact operators.

**Essential Norm Lemma.** \( \| S_f \|_{\text{ess}} \leq 2 \quad (\leq 1) \)

**Weak Convergence Lemma.** Let a sequence \( f_n \) of vectors in a Hilbert space converge weakly to a vector \( f \). Then

\[
\limsup_{n \to \infty} \| f_n \|^2 = \| f \|^2 + \limsup_{n \to \infty} \| f - f_n \|^2.
\]

**Proof.** Let \( P \) be the orthogonal projection onto the linear span of \( f \). The condition \( f_n \overset{w}{\to} f \) implies \( \| f - P f_n \| \to 0 \). The lemma follows immediately.

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Suppose \( \exists f_n \in X_n, \| f_n \| = 1 \) such that

\[
|f_n(t)| \leq 0.1 \cdot |(S_n f_n)(t)| \quad \forall t \in \Gamma.
\]

(equiv. \( \mu \geq 10 \))

Taking a subsequence, if necessary, we can assume weak convergence \( f_n \overset{w}{\to} f \).

Consider the simple case \( f = 0 \). The condition \( |f_n(t)| \leq 0.1 \cdot |(S_n f)(t)| \) implies that

\[
\| f_n \| \leq 0.1 \| P_n S f_n \| \leq 0.1 \| S f_n \|.
\]

Since \( \| S \|_{\text{ess}} \leq 2 \), there exists a compact operator \( K \) such that \( \| S - K \| \leq 3 \). Therefore

\[
1 = \limsup_{n \to \infty} \| f_n \| \leq 0.1 \limsup_{n \to \infty} \| S f_n \|
\]

\[
= 0.1 \limsup_{n \to \infty} \| (S - K) f_n \| \leq 0.3,
\]

and we get a contradiction!
Suppose now that $f_n \xrightarrow{w} f \neq 0$.

We had $|f_n(t)| \leq 0.1 \cdot |(S_n f)(t)| \quad \forall t \in \Gamma$. (assumption that $\mu \geq 10$)

\[ 10 \limsup_{n \to \infty} \| E(\Delta) f_n \| \leq \limsup_{n \to \infty} \| E(\Delta) S f_n \| \]
\[ \leq \limsup_{n \to \infty} \| E(\Delta) S f \| + \limsup_{n \to \infty} \| E(\Delta) (f - f_n) \| \]

Let $g_n := f - f_n$. Clearly $f_n \to f$ weakly, and $\limsup_{n} \| g_n \| \leq 1$.

In the following lemma the sequence $g_n \xrightarrow{w} 0$ is supposed to be fixed.

**Lemma.** For any point $\tau \in \Gamma$ and any positive $\varepsilon, \delta$, there exists an arc $\Delta \ni \tau$, $|\Delta| \leq \varepsilon$ such that

\[ \limsup_{n \to \infty} \| E(\Delta) S g_n \| \leq 2 \limsup_{n \to \infty} \| E(\Delta) g_n \| + |\Delta| \cdot \delta. \]

Follows from the Localization Property.

Completing the proof. By Lemma about weak convergence

\[ \limsup_{n \to \infty} \| E(\Delta) f_n \| = \sqrt{\| E(\Delta) f \|^2 + \limsup_{n \to \infty} \| E(\Delta) g_n \|^2} \]
\[ \geq \frac{1}{\sqrt{2}} \left( \| E(\Delta) f \| + \limsup_{n \to \infty} \| E(\Delta) g_n \| \right), \]

where $g_n = f - f_n$.

Therefore, for any point $\tau \in \Gamma$ and any $\varepsilon > 0$, (put $\delta = 1$) $\exists$ arc $\Delta \ni \tau$, $|\Delta| < \varepsilon$, such that

\[ \| E(\Delta) f \| + \limsup_{n \to \infty} \| E(\Delta) g_n \| \leq \sqrt{2} \limsup_{n \to \infty} \| E(\Delta) f_n \| \leq \frac{\sqrt{2}}{10} \limsup_{n \to \infty} \| E(\Delta) S f_n \| \]
\[ \leq \frac{\sqrt{2}}{10} \left( \| E(\Delta) S f \| + \limsup_{n \to \infty} \| E(\Delta) S g_n \| \right) \]
\[ \leq \frac{\sqrt{2}}{10} \left( \| E(\Delta) S f \| + 2 \limsup_{n \to \infty} \| E(\Delta) g_n \| + |\Delta| \right) \]

so

\[ \| E(\Delta) f \| \leq \frac{\sqrt{2}}{10} \left( \| E(\Delta) S f \| + |\Delta| \right). \]

By Lebesgue density Theorem

\[ |f(\tau)| \leq \frac{\sqrt{2}}{10} |(S f)(\tau)| \]

Therefore $\exists \varphi \in L^\infty$, $\| \varphi \|_\infty \leq \sqrt{2}/10$ such that

\[ f = \varphi S f. \]

But the operator $I - \varphi S$ is invertible, so we got a contradiction.
What about upper bounds for other structured norms?

If $\mathbb{U}$ = diagonal, LTV (or NLTI, or NLTV), and the scaling operators $D \in \mathcal{D}$ are constant diagonal matrices, then

$$SN_{\mathbb{U}}(G) = \overline{SN}_{\mathbb{U}}(G) := \inf_{D \in \mathcal{D}} \|DGD^{-1}\|$$

**Reason:** $S$-procedure works for time invariant (shift invariant) forms. (A. Megretskii, S. Treil, 1990)

**Theorem** Let $\sigma_k$ be time invariant quadratic forms, such that

$$\sigma_k[x] \geq 0, \quad \forall k = 1, \ldots, n \implies \sigma_0[x] \geq 0,$$

and such that

$$\exists x_0, \quad \sigma_k[x_0] > 0, \quad k = 1, \ldots, n.$$

Then there exist $\tau_k \geq 0$ such that

$$\sigma_0[x] - \sum_{k=1}^{n} \tau_k \sigma_k[x] \geq 0 \quad \forall x.$$

**Open problems**

1. Find an “algebraic” proof of the result, for example working directly with discrete analogues of SIO, not discretizing continuous SIO

2. Find an asymptotic for $\overline{\mu}/\mu$:
   - **Known:** $\overline{\mu}/\mu \leq Cn$ (A. Megretskii)
   - **Conjecture:** $\overline{\mu}/\mu \leq C \log n$ and the asymptotic is sharp

3. “Practical” estimates, say for $4 \leq n \leq 50$

4. Let $G$ be a multiplication by $H^\infty$ function (scalar, rational, with real coefficients), and let the class of uncertainties $\mathbb{U}$ coincide with all real Fourier multipliers,

$$\Delta f = \sum a_k \hat{f}(k) z^k, \quad a_k \in \mathbb{R}.$$ 

Is $SN_{\mathbb{U}}(G)/\|G\|$ bounded?

If we allow complex $a_k$, then trivially

$$SN_{\mathbb{U}}(G)/\|G\| = 1,$$

but complex $a_k$ do not make any sense in control.