$\| x - \sum_1^n \alpha_k e_k \| \to \min$ is attained at $\alpha_k = (x, e_k)$.

Pf: Define $L = \mathcal{L}(e_1, \dots, e_n) = \{ \sum_{k=1}^n \alpha_k e_k : \alpha_k \in \mathbb{F} \}$.

$y = \sum_{k=1}^n (x, e_k) e_k$. $(x-y, e_j) = (x, e_j) - (x, e_j)(e_j, e_j) = 0$.

$x - y \perp e_j$ $(1 \le j \le n)$. $\therefore$, $x - y \perp z$ $\forall z \in L$. $x - y \perp L$.

Lem: If L is a subspace of $\mathcal{H}$, $x \in \mathcal{H}$, $y \in L$, s.t. $x - y \perp L$.
Then, $\forall z \in L$, $z \neq y$, $\|x - z\| > \|x - y\|$.

Pf: $\|x - z\|^2 = \|(x-y) + (y-z)\|^2 = \|x - y\|^2 + \underbrace{\|y - z\|^2}_{>0} > \|x - y\|^2$.

Rmk: $y \in L$ s.t. $x - y \perp L$ is unique (if it exists).

$y = P_L x$.

If $\dim L < \infty$, $P_L x$ exists. $P_L x = \sum_1^n (x, e_k) e_k$. $\{e_k\}_1^n$ = ONB for L

If L is separable (in particular, if $\mathcal{H}$ is separable) and $\{e_n\}_1^\infty$ is ONB in L, then $P_L x = \sum_{k=1}^\infty (x, e_k) e_k$.

Lem (Bessel's Inequality): If $\{e_n\}_1^\infty$ is ONB, then
$\sum |(x, e_n)|^2 \le \|x\|^2$.

Pf: Let $L_n = \mathcal{L}(e_1, \dots, e_n)$. Then, $P_{L_n} x = \sum_1^n (x, e_k) e_k$.
$\|P_{L_n} x\|^2 = \sum_1^n |(x, e_k)|^2$. $\|x\|^2 = \|P_{L_n} x\|^2 + \|x - P_{L_n} x\|^2$. ...

Therefore, $\|P_{L_n} x\|^2 \leq \|x\|^2$. $\Longrightarrow \sum_{k=1}^{n} |(x, e_n)|^2 \leq \|x\|$.

Let $n \to \infty$. Then, $\sum_{k=1}^{\infty} |(x, e_k)|^2 \leq \|x\|$.

Bessel's Inequality $\Longrightarrow \sum_{k=1}^{\infty} (x, e_k) e_k$ converges.

$x - \sum_{1}^{\infty} (x, e_k) e_k \perp e_n \; \forall n$. $x - y \perp \sum \alpha_k e_k \Rightarrow x - y \perp L$

because $\sum_{\text{finite}} \alpha_k e_k$ are dense in L.

<u>Thm</u>: Let $M \subset \mathcal{H}$ be convex and closed, $x \in \mathcal{H}$. Then, $\exists!\, y \in M$

s.t. $\|x - y\| = \operatorname{dist}(x, M) = \inf\{\|x - y\| : y \in M\}$.

<u>Pf</u>: $\exists\, y_n \in M$ s.t. $\|x - y_n\| \to d$. Goal: $\{y_n\}_1^{\infty}$ is Cauchy

$\forall \epsilon > 0 \;\; \exists N$ s.t. $\forall n > N \;\; d \leq \|x - y_n\|^2 < d^2 + \epsilon^2/4$.

$m, n > N$. Use parallelogram identity for $x - y_n$ and $x - y_m$:

$$2(\|x - y_n\|^2 + \|x - y_m\|^2) = \underbrace{\|2(x - \underbrace{\tfrac{y_n + y_m}{2}}_{\in M})\|^2}_{\geq 4d^2} + \|y_n - y_m\|^2.$$

$$2(\underbrace{\|x - y_n\|^2}_{< d^2 + \frac{\epsilon^2}{4}} + \underbrace{\|x - y_m\|^2}_{< d^2 + \frac{\epsilon^2}{4}}) < 4d^2 + \epsilon^2.$$

$\therefore$, $\|y_n - y_m\|^2 < \epsilon^2$, or $\|y_n - y_m\| < \epsilon$. $\therefore$, $\{y_n\}_1^{\infty}$ is Cauchy,

so $\exists\, y = \lim_{n \to \infty} y_n$, $y \in M$ because M is closed.

$\|x - y\| = \lim_{n \to \infty} \|x - y_n\| = d$.

Why unique? Suppose $y \neq z \in M$, $\|x-y\| = \|x-z\| = d$.

$y_{2n} = y$, $y_{2n+1} = z$. $\lim_{n\to\infty} \|x-y_n\| = d$, so $\exists \lim y_n$. But $\{y_n\}$ is not Cauchy! $\divideontimes$.

<u>Rmk</u>: If M is a subspace, then uniqueness was already proved. If M is a subspace, then y s.t. $\|x-y\| = d = \mathrm{dist}(x, M)$ is equal to $P_M x$.

<u>Pf</u>: $M = L =$ closed subspace. Want $x - y \perp L$. Take arbitrary $z \in L$, $\|z\| = 1$. If $(x-y, z) \neq 0$, then $z_0 = (x-y, z)z = P_{\mathcal{L}(z)}(x-y)$.

$\Rightarrow \|x-y-z_0\| < \|x-y\| = d$. But $y + z_0 \in L$, so $\|x-(y+z_0)\| < d$. $\divideontimes$.