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Lemma: $\operatorname{dist}(x, L) = \max\{ |f(x)| : f \in X^*, \|f\| \le 1, f|_L = 0 \}$

Pf: $y \in L$ consider $|f(x)| = |f(x-y)| \le \|f\| \|x-y\| \le \|x-y\|$

choose $y \in L$ infimum $\Rightarrow |f(x)| \le \operatorname{dist}(x, L)$

Let $L_1 = \operatorname{span}(x, L) = \alpha\vec{x} + y \quad y \in L \quad \alpha \in \mathbb{F}$

$f_0(\alpha\vec{x} + \vec{y}) = \alpha d$

$|f_0(\alpha\vec{x} + \vec{y})| = |\alpha| |d| = \operatorname{dist}(\alpha\vec{x}, L) \le \|\alpha\vec{x} + \vec{y}\|$

so $\|f_0\| \le 1$

Take y_n s.t. $\|\alpha\vec{x} + \vec{y}_n\| \to |\alpha| d$

$|\alpha| d = |f(\alpha\vec{x} + y_n)| \le \|\alpha\vec{x} + \vec{y}_n\| \to |\alpha| d$

$\forall \varepsilon > 0 \quad \exists N$ s.t. $\forall n > N$

$\|f(\alpha\vec{x} + \vec{y}_n)\| \ge \|\alpha x + y_n\| - \varepsilon$

$\Rightarrow \|f_0\| = 1$

Extend f_0 to f by Hahn-Banach

$f|_L = 0 \qquad \|f\| = 1 \qquad f(x) = f_0(x) = d$

Baby Example: $f \in L^p(0,1) \quad 1 \le p < \infty$

What is $\inf\{ \|f-g\|_p : g \in L^p, \int_0^1 g(x)\,dx = 0 \}$?

Take $\varphi \in L^{p'}$ s.t. $\int_0^1 \varphi g\,dx = 0$

$\varphi(g) = \int_0^1 1 \cdot g(x)\,dx$

$\inf = \left| \int_0^1 f(x)\,dx \right|$

Minimal Systems, Biorthogonal (Dual) Systems and Bases

Def: $\{x_n\}_1^\infty$ $x_n \in \mathfrak{X}$ $\{x_n\}_1^\infty$ is Linearly Independent if $\{x_n\}_1^N$ is L.I. $\forall$ $N < \infty$.

(If $\sum_{\text{finite}} a_n \vec{x}_n = 0$ then all $a_n = 0$)

Def: $\{x_n\}_1^\infty$ is called minimal if $\forall n$, $\operatorname{dist}(x_n, \mathcal{L}(x_k : k \neq n)) > 0$.

Note: minimal $\Rightarrow$ linear independence

Thm: $\{x_n\}$ is minimal iff $\exists$ $\{x_n'\}$ $x_n' \in \mathfrak{X}^*$ s.t. $\langle x_n, x_k' \rangle = \delta_{n,k}$

Def: $\{x_k'\}$ is called dual (biorthogonal) to $\{x_k\}$.

$$\operatorname{dist}(x_n, \mathcal{L}(x_k : k \neq n)) = \max\left\{ |\langle x_n, x^* \rangle| : x^* \in \mathfrak{X}^*_1 \ \|x^*\| = 1 \ \langle x_k, x^* \rangle = 0 \ \forall k \neq n \right\}$$

then $x_n' = \frac{1}{d} x^*$

Def: $\{x_n\}$ is called a basis if $\forall x \in \mathfrak{X}$ $\exists!$ representation $x = \sum a_k x_k$ and the series converges.

Def: $\{x_n\}$ is called an unconditional basis if $\forall x$ $\exists!$ representation $x = \sum_1^\infty a_k x_k$ and the series converges unconditionally (under any reordering)

Ex: $L^p_{2\pi} \simeq L^p_{(\mathbb{T})}$ $\qquad L^p_{2\pi}(\omega)$ $\qquad \|f\|_{L^p(\omega)} = \int_0^{2\pi} |f(x)|^p \omega(x) \frac{dx}{2\pi}$

Claim $(L^p(\omega))^* = L^{p'}(\frac{1}{\omega})$ in particular $(L^2(\omega))^* = L^2(\frac{1}{\omega})$

$$\langle f, g \rangle = \int f g \frac{dx}{2\pi}$$

Note that $L^2(\omega)$ is a Hilbert space