1. Compute the characteristic function of the Bernoulli random variable $\xi$, $P(\xi = 1) = p$, $P(\xi = 0) = q = 1 - p$. Use this result to compute the characteristic function of the binomial random variable $X$

$$
P(X = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \ldots, n.$$

$$
f_\xi(t) = E(e^{it\xi}) = q^t + pe^{it}.$$ 

$$
X = \sum_{i=1}^{n} \xi_i \quad \xi_i \text{ - independent Bernoulli}$$

The characteristic function of a sum of independent random variables is the product of the characteristic functions,

$$
\text{so } f_X(t) = (q + pe^{it})^n$$
2. For a Markov chain with the transition matrix

\[
\begin{pmatrix}
q & p & 0 & 0 & 0 \\
q & 0 & p & 0 & 0 \\
0 & q & 0 & p & 0 \\
0 & 0 & 0 & q & p \\
0 & 0 & 0 & 0 & q \\
\end{pmatrix}, \quad p + q = 1, \ p, q > 0.
\]

Draw the graph and determine which states are transient and which are persistent.

States 1, 2, 3 are transient, because with non-zero probabilities we get to state 4 and then never return back.

States 4, 5, 6 are persistent because if we are in one of these states, we never leave, and all states \((4, 5, 6)\) accessible from each other (cycle \(4 \rightarrow 5 \rightarrow 6 \rightarrow 5\)).
3. For a Markov chain with the transition matrix

\[
\begin{pmatrix}
0 & p & 0 & 0 & q \\
p & 0 & p & 0 & 0 \\
0 & q & 0 & p & 0 \\
0 & 0 & q & 0 & p \\
p & 0 & 0 & q & 0 \\
\end{pmatrix}
\]

\(p + q = 1, p, q > 0,\)

determine whether limiting probabilities exist or not. If limiting probabilities exist, compute them.

**Hint:** The answer does not depend on \(p, q\). Using this information you can easily guess a solution, especially if you notice that all columns add up to 1.

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**Diagram:***

1. Chain is persistent
   - Cycle 1 → 2 → 3 → 4 → 5

2. Each state is aperiodic:
   - State 1: 1 → 2 → 1, 2 steps
   - 1 → 2 → 3 → 4 → 5, 5 steps
   - \(\gcd(2, 5) = 1\)
   - For all other states, similarly, no difference with 1

Easy to check:

\[\begin{pmatrix}
1, 1, 1, 1, 1 \\
1, 1, 1, 1, 1 \\
0, 0, q, 0, 0 \\
0, 0, q, 0, 0 \\
p, 0, 0, q, 0 \\
\end{pmatrix}\]

\[p_k = \frac{1}{5}\text{ in a solution}\]
4. The arrival of the customers to the service desk is described by a Poisson process with parameter \( \lambda \). Suppose there is only one clerk at the desk, and let the time \( r \) that is required for the clerk to service the customer is an exponentially distributed random variable with parameter \( \mu \). If a customer finds the clerk busy, he waits in line; however, if there are already 2 customers waiting (in addition to one being served), the customer just leaves.

(a) Write down the Kolmogorov equations for the probabilities \( p_k(t) \) (the state is the number of customers at the service desk (waiting and being served)).

(b) For \( \lambda = \mu \) find out limiting probabilities; be sure to explain why they exist.

\[
\begin{align*}
p'_0(t) &= -\lambda p_0 + \mu p_1 \\
p'_1(t) &= -\lambda p_1 - \mu p_1 + \lambda p_2 + \lambda p_0 \\
p'_2(t) &= -\lambda p_2 + \mu p_2 + \lambda p_1 + \mu p_3 \\
p'_3(t) &= -\mu p_3 + \lambda p_2
\end{align*}
\]

All states interconnected (6\( \to \)2\( \to \)3\( \to \)2\( \to \)1\( \to \)0)

So limiting probabilities exist.

Find limiting prof for \( \lambda = \mu \)?

\[
\begin{align*}
0 &= -\lambda p_0 + \lambda p_1 \\
0 &= -2\lambda p_1 + \lambda p_2 + \lambda p_0 \\
0 &= -2\lambda p_2 + \lambda p_3 + \lambda p_1 \\
0 &= -\lambda p_3 + \lambda p_2
\end{align*}
\]

So \( p_0 = p_1 = p_2 = p_3 \ldots \)

\[
\begin{align*}
p_1 &= p_0 \\
o &= (-2p_0 + p_1 + p_0)\lambda \Rightarrow p_2 = p_0 \\
o &= (-2p_0 + p_1 + p_3)\lambda \Rightarrow p_3 = p_0 \\
0 &= (-2p_2 + p_3)\lambda \Rightarrow p_3 = p_6
\end{align*}
\]

so condition \( \lambda = p_k = 1 \) gives us

\[
p_k = \frac{1}{4} \quad k = 0, 1, 2, 3
\]

Limiting probabilities
5. A die is rolled repeatedly. Show that the number $\xi(n)$ of sixes in $n$ rolls is a Markov chain and find the transition matrix. Can you tell if the states are transient or persistent?

$$P_{ij} = \begin{cases} \frac{1}{6}, & j = i+1 \\ \frac{5}{6}, & j = i \\ 0, & \text{otherwise} \end{cases}$$

All states are transient, because with prob $\frac{5}{6} > 0$ we go from state $i$ to state $i+1$ and then never return back to $i$. 