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Subharmonicity and Laplacian, Green's Functions.

Green's Formulas

Divergence Thm: $\int_{\partial G} \vec{F}\cdot \underset{\text{outer normal}}{\vec{n}}\, ds = \iint_G \operatorname{div} \vec{F}\, dx\, dy$ (particular case of Stokes')

(Stokes' Thm: $\int_{\partial\Omega} \omega = \int_\Omega d\omega$)

Assume ∂G is C^1 or P-C^1.

$F \in C^1(\mathrm{Cl}\, G)$. G bounded region.

Let $u, v \in C^1(\mathrm{Cl}\, G)$

$$\vec{F} = u\nabla v - v\nabla u$$
$$= u\begin{pmatrix} v_x \\ v_y \end{pmatrix} - v\begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

$$\vec{F}\cdot\vec{n} = u\,\frac{\partial v}{\partial n} - v\,\frac{\partial u}{\partial n}$$

$$\operatorname{div} F = u\,\Delta v - v\,\Delta u$$

$$\frac{\partial}{\partial x}(F_1) = u_x v_x + u v_{xx} - (v_x u_x + v u_{xx})$$
$$= u v_{xx} - v u_{xx}$$

$$\frac{\partial}{\partial y}(F_2) = u_y v_y + u v_{yy} - (v_y u_y + v u_{yy})$$
$$= u v_{yy} - v u_{yy}$$

$$\left.\int_{\partial G}\left(u\,\frac{\partial v}{\partial n} - v\,\frac{\partial u}{\partial n}\right) ds = \iint_G (u\Delta v - v\Delta u)\, dx\, dy \right\}\ 1^{st} \text{ Green Formula}$$

Note: In complex variables, $\omega = u\,\frac{\partial v}{\partial z}\, dz - v\,\frac{\partial u}{\partial \bar{z}}\, d\bar{z}$. Then apply Stokes' Thm.

$$G = G_\varepsilon = \{z : \varepsilon < |z| < 1\}$$

Let $u = \log\frac{1}{|z|}$ Harmonic in G_ε. Let v arbitrary $\in C^2(\mathrm{Cl}\,\mathbb{D})$

$$\iint_{G_\varepsilon} \log\frac{1}{|z|}\,\Delta v\, dx\, dy = \int_{\partial G_\varepsilon}\left(u\,\frac{\partial v}{\partial n} - v\,\frac{\partial u}{\partial n}\right) ds.$$

[Note $\frac{\partial \log|z|^{-1}}{\partial n} = \frac{\partial \log(1/r)}{\partial r} = -\frac{1}{r}$.]

$$= \int_{|z|=1} v\,|z| + \underbrace{\int_{|z|=\varepsilon} \log\frac{1}{|z|}\,\frac{\partial v}{\partial n}\,|dz|}_{\varepsilon\log(1/\varepsilon)\to 0} + \underbrace{\int_{|z|=\varepsilon} v(z)\,\frac{1}{r}\,|dz|}_{\to 2\pi v(0)}$$

Letting $\varepsilon \to 0$

$$\iint_{\mathbb{D}} \Delta v \log\left(\frac{1}{|z|}\right) dA(z) = \int_{|z|=1} v(z)\,|dz| - 2\pi\, v(0)$$

$$\int_{\partial \mathbb{D}} v(z) \frac{|dz|}{2\pi} - v(0) = \frac{1}{2\pi} \iint_{\mathbb{D}} \Delta v \cdot \log \frac{1}{|z|}\, dA(z) \Bigg\} \text{ 2nd Green Formula}$$

Note: Mean Value Property for harmonic functions follows directly from this formula.

<u>Cor</u> $u \in C^2(\Omega)$ is SH $\iff$ $\Delta v \geq 0$.

~~Consider $\Delta v = \varphi$~~

<u>Rmk</u> Let Ω a region, $z_0 \in \Omega$. Let $\Omega_\varepsilon = \Omega \setminus D_{z_0,\varepsilon}$.

Let $G \in \text{Harm}(\Omega \setminus \{z_0\})$, $G \in C^1(\partial\Omega)$

Near z_0, G behaves like $\log \frac{1}{|z-z_0|}$

$G(z) - \log \frac{1}{|z-z_0|}$ bdd.

$$-\int_{\partial\Omega} v(z) \frac{\partial G}{\partial n} \frac{|dz|}{2\pi} - v(z_0) = \frac{1}{2\pi} \iint_\Omega \Delta v \quad G(z)\, dx\,dy$$

$G = G_{z_0}$

<u>Ex</u> If $\Omega = \mathbb{D}$. $G_{z_0} = \log \left| \frac{1-\bar{z}_0 z}{z - z_0} \right|$

Taking normal derivative, we get Poisson kernel.

$$-\frac{\partial}{\partial n}\left(\log \left|\frac{1-\bar{z}_0 z}{z-z_0}\right|\right) = \frac{1-|z_0|^2}{|1-\bar{z}_0 z|^2}.$$

$$\mathcal{H} v(z_0) - v(z_0) = \iint_{\mathbb{D}} \Delta v(z) \log\left|\frac{1-\bar{z}_0 z}{z-z_0}\right| dx\,dy$$

harmonic extension

Note: This can be obtained from 2nd Green Formula by change of variables.

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Let $\Delta v = \varphi$, $v|_{\partial\Omega} = 0$